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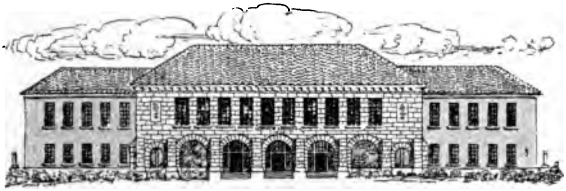

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PLANE GEOMETRY

BY

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AUTHOR'S NOTE

In adding to the number of text-books on Geometry the author aims not so much to provide new material for college preparatory work in the subject, as to furnish if possible to the student a more suggestive method of study and a more graphic form of written demonstration.

Our American text-books have practically agreed upon the essential theorems in geometric demonstration, and novelties in subject-matter are unnecessary. It is believed, however, that the subject may yet acquire a new attractiveness and be more promptly mastered if a few important principles are magnified.

Experience shows that a student's attention should be directed to the geometric relation itself, rather than to the description of the relation — to the properties of the figures used in illustrating the geometric fact, rather than to verbal discussions of hypothesis and conclusion. Valuable as the argumentative drill may be, the study of Geometry has a mathematical value which exceeds even the charm of its logic.

Geometric figures should speak for themselves — carry their own hypotheses and suggest their own inquiries. By a simple method of marking the lines many of the equalities between the parts of figures are shown; others will readily suggest themselves. A student should not only appreciate these symbols but should use them constantly, so that in meeting new problems he may instinctively transfer to the figure as much as is possible of the hypothesis, and from the data thus graphically before him proceed to the discussion and the solution.

The propositions of Interpretation given on pages 52–56, 84–86, 116–118, 180–182 illustrate the emphasis laid upon this feature of the text.

In the written demonstration attention is paid to the logical arrangement of major and minor steps; the important or principal steps are placed in columns by themselves, and the secondary steps, or supporting reasons, are placed in brackets beside the primary steps. These minor steps are often omitted in the later books and should be supplied when necessary by the student. The chain of reasoning is kept distinct, and is condensed in form and phraseology, even at the sacrifice of literary form.

The use of such methods has satisfied the author that the concentration of the student's attention upon the few main steps of proof in each demonstration, simplifies the mastery of the principles involved, and in written work facilitates such examination and criticism by a class, and by the instructor, as is rarely possible in a full-text, disarranged form of proof.

Lack of confidence is a common complaint among students in attacking and solving original exercises. It has been the author's privilege to find several hundred pupils acquiring confidence and facility in original demonstration, largely through the use of graphic figures and a condensed, well-arranged form of written proof.

The exercises have been selected and devised with much care; the list furnishes no mathematical puzzles and no problems beyond the grasp of beginners in the study of Geometry.

A brief discussion of Logarithms as required for the solution of simple Geometric problems is appended to the text, — also tables of Logarithms of Numbers, and Metric Tables.

The attention of teachers is respectfully directed to the features above mentioned, to the topical summary on page 172, and to the college examination papers on page 217; criticism is cordially invited.

Opportunity is here taken to express to my colleague, Mr. George T. Eaton, my appreciation of his assistance in the preparation of this work.

ANDOVER, MASS., April, 1896.

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GEOMETRY

DEFINITIONS

1. A **Geometric Solid** is a portion of space. The space occupied by a material or physical solid is a geometric solid.

2. The **Surface** of a solid is its faces; by its surface the solid is separated from the surrounding space.

3. The intersection of two surfaces is called a **line.**

4. The intersection of two lines is called a **point.**

5. A point moving generates a line; a line moving, not in its own direction, generates a surface; a surface moving, not in its own plane or upon itself, generates a solid.

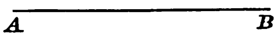
6. The **Dimensions** of a solid are its *length, breadth,* and *thickness.* Extent is measured in one or more of these dimensions.

7. The **Science of Geometry** treats of extension, making use of geometrical figures; the properties, measurement, and construction of these figures are considered.

8. A **Point** has position, but not extent.

9. A **Line** has length, without breadth or thickness.

10. A **Straight line,** or **Right line,** is one of constant direction, or rather of two directions exactly opposite; as *AB* or *BA.*



11. A **Curved line** is one of continually changing direction; as *CD*.

12. A **Surface** has length and breadth, but no thickness.

13. A **Plane** is such a surface that a straight line joining any two of its points lies wholly in the surface.

14. A **Solid** has length, breadth, and thickness.

15. A **Theorem** is a statement of a truth to be proved. The *hypothesis* of a theorem is the conditional part of the statement, that which is assumed to be true; the *conclusion* is that which is to be proved.

16. A **Problem** is a statement of something to be done.

17. A **Proposition** is a general term for either a theorem or a problem.

18. A **Corollary** is a secondary or derived theorem, suggested by the theorem to which it is attached, and is usually proved after the same manner as the theorem itself.

19. A **Scholium** is a remark appended to a proposition.

20. An **Axiom** is a truth assumed as self-evident.

21. AXIOMS

1. Things which are equal to the same thing are equal to each other.

It follows that, *a.* Things equal to equal things are equal to each other. *b.* Things that may coincide throughout are equal. *c.* In an equation, or in any other algebraic expression, equals may be substituted for equals.

2. The whole is equal to the sum of all its parts, and is therefore greater than any of its parts.

3. Equals added to equals give equals.

4. Equals subtracted from equals give equals.

5. Equals multiplied by equals give equals.
6. Equals divided by equals give equals.
7. Like powers and like roots of equals are equal.
8. A straight line is the shortest distance between two points.

22. A **Postulate** is an assumption that a problem is possible.

23. POSTULATES

1. It is assumed that a straight line can be drawn between any two points, or can be produced to any required length.
2. It is assumed that a geometric figure, or portion of a figure, can be moved without changing the relations of its parts to each other.

24. ORDER OF WRITTEN PROOF

A proposition may be divided into five parts: 1. Theorem (or Problem); 2. Application; 3. Construction; 4. Demonstration; 5. Conclusion. These may be followed by Corollaries and Scholia.

25. The **Theorem** (or **Problem**) is the *general* statement of the proposition to be considered.

26. The **Application** is the *particular* statement of the proposition as applied to the figure which is used in the process of proof.

27. The **Construction** is the drawing of auxiliary lines or figures used in the proof, or the placing of figures in any desired position.

28. The **Demonstration** is the *proof* proper, and consists of the logical reasoning employed to establish the truth of the theorem.

29. The **Conclusion** is a final statement of the theorem as proved, and in written demonstration is often omitted.

30. Plane Geometry treats of figures whose elements are all in the same plane.

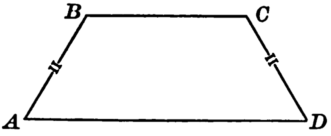
31. Solid Geometry treats of figures whose elements are not all in the same plane.

32. Plane figures are sometimes obtained by passing a plane through a geometric solid, and such figures are *section views* of the solid.

33. SIGNS AND ABBREVIATIONS

+	*plus.*	∥	*parallel.*
−	*minus.*	∥s	*parallels.*
×	*multiplied by.*	▱	*parallelogram.*
÷	*divided by.*	▱s	*parallelograms.*
=	*equal.*	⊙	*circle.*
≡	*equivalent.*	⊙s	*circles.*
<	*less than.*	⌒	*arc.*
>	*greater than.*	⌒s	*arcs.*
△	*triangle.*	theo.	*theorem.*
△s	*triangles.*	appl.	*application.*
∠	*angle.*	cons.	*construction.*
∠s	*angles.*	cor.	*corollary.*
rt. ∠	*right angle.*	ax.	*axiom.*
rt. ∠s	*right angles.*	sch.	*scholium.*
⊥	*perpendicular.*	hyp.	*hypothesis.*
⊥s	*perpendiculars.*		etc.

34. The same letter or figure used by repetition to designate two or more angles indicates that the angles are equal, and usually that they are equal by hypothesis in the discussion of any theorem. Such angles may be designated thus: x, x', x'', etc.



The same mark, or the same number of small cross lines placed upon two lines in a figure indicates that the two lines are equal, usually by hypothesis, as

$$AB = CD.$$

Short parallels placed as in Fig. 1 show that *AB* is ∥ *DC*, *i.e.* *ABCD* is a trapezoid (for definition see p. 38). *A* and *D* are shown to be rt. ∠s, and it is evident that *AD* is bisected, also *BC*.

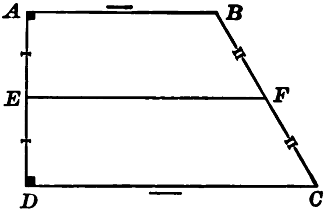


Fig. 1.

In Fig. 2 *ABC* is, by means of the mark of equality, shown to be an isosceles △ (for definition see p. 24). Also, the values given to the ∠s of the isosceles △ show that the ∠ at *B* is four times either of the ∠s *A* and *C*, and of course twice their sum.

DC is ⊥ *AC* and is met by *AB* produced.

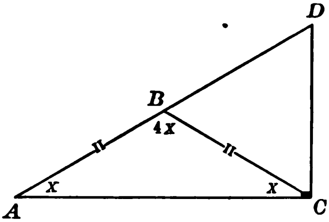
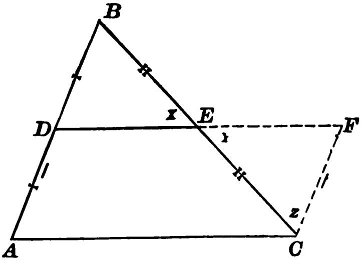


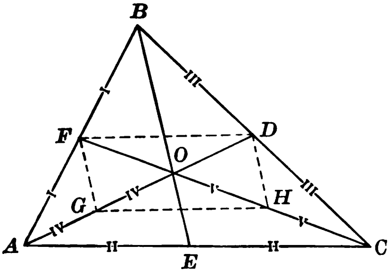
Fig. 2.

The figures thus marked are graphic pictures of geometric relations, and from them deductions may be made.

Observe the bisection of lines, and the consequent equality of parts of lines, indicated in the accompanying figures:

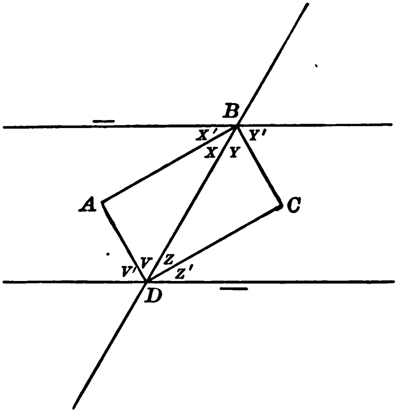


(see section 121)

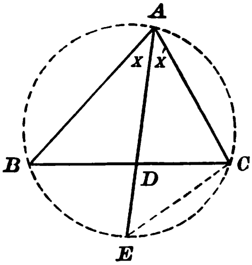


(see section 118)

The equality of angles is shown in such figures as the following:



(see ex. 59, p. 54)



(see section 223)

NOTE.—The student should early accustom himself to the use of these and other symbols of equality. For example, a *double* cross may indicate a *common* line or angle; that is, a line or angle common to two distinct geometric figures, as the angle *A*, in the triangles *ABC* and *ADE*, in section 255.

PLANE GEOMETRY

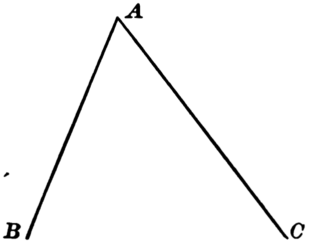
BOOK I

DEFINITIONS

The Point

35. The position of a point is determined by its direction and distance from a known point; or its direction from two known points, provided the three points are not in the same straight line.

Thus, the position of A is known, if its direction and distance from B are known, or the position of A is determined if its direction from both B and C is known; BA and CA representing the direction of the point A from B and C respectively, A lies at the intersection of these two lines.



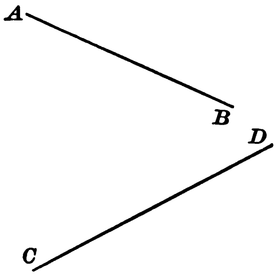
The Straight Line

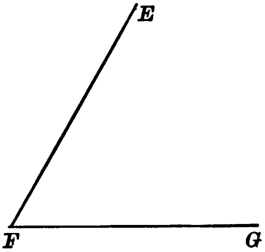
36. A straight line is determined in position by any two of its points, or by one point and its direction.

37. The **Origin** of a line is the point from which the line is supposed to be drawn.

ANGLES

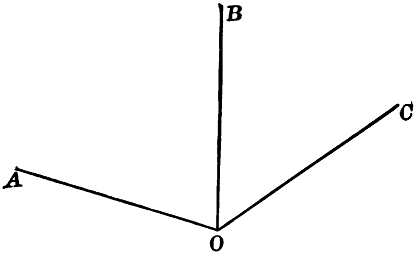
38. An angle is the difference in direction of two lines. If the lines meet, the intersection is called the **vertex**; the lines are called **sides**. The lines *AB* and *CD* form an angle with each other; *F* is the vertex of the angle formed by the sides *FE* and *FG*.



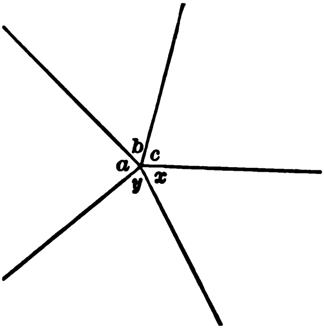


If *FE* is regarded as having at first coincided with *FG*, and then revolved about *F* as a pivot, *GFE* is called an **angle of rotation.**

When lines meet forming but one angle, the angle may be designated by the single letter at the vertex, but when several angles have a common vertex, each angle is designated by three letters, the middle letter showing the vertex, as the angles *AOB*, *BOC*.

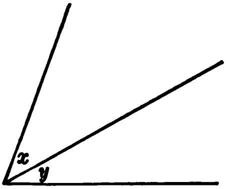


39. Angles may also be designated by small letters written near the vertex and between the sides, as the angles *a*, *b*, *c*, *x*, *y*.

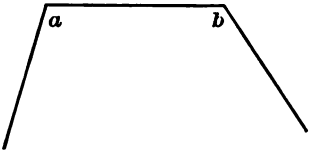


40. The magnitude of an angle depends solely upon the amount of divergence, that is, the difference in direction, of its sides, and is independent of their length.

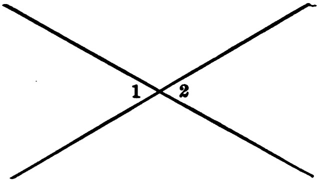
41. **Adjacent** angles are those which have the same vertex and a common side lying between the other two sides, as *x* and *y*.



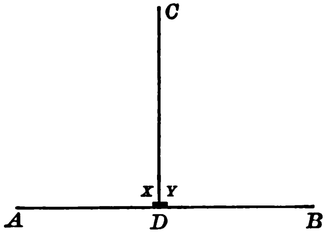
42. **Consecutive** angles are those which have different vertices and a common side between them, as *a* and *b*.



43. **Vertical** angles are those which have the same vertex and so situated that the sides of one are the prolongation of the sides of the other, as 1 and 2.



44. Two angles are equal when they may be made to coincide throughout, vertex with vertex and side with side.



45. When one straight line meets another so as to make the adjacent angles equal, each of the angles is called a **right angle**, and the lines are said to be **perpendicular** to each other. If $X = Y$, each is a right angle, and AB and CD are $\perp$ to each other.

46. The unit of angular measurement is the right angle. For convenience, a right angle is divided into ninety equal parts, called **degrees.**

A degree is subdivided into sixty equal parts called **minutes,** and a minute into sixty equal parts called **seconds.**

An angle, thus measured, is expressed by means of symbols, as $67° 32' 45''$, and is read 67 degrees, 32 minutes, and 45 seconds.

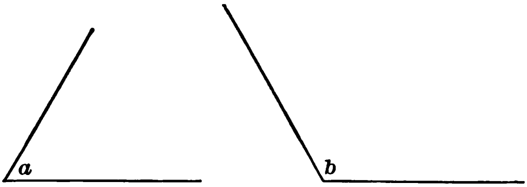
47. The last three sections may be summarized as follows:

a. **Coincidence** is the basis of *equality.*

b. **Equality** is the basis of *perpendicularity* and of the definition of *right angles.*

c. The **right** angle is the *unit angle.*

48. An **acute** angle is one which is less than a right angle, as *a.*



49. An **obtuse** angle is one which is greater than a right angle, as *b.*

50. All angles which are not right angles are **oblique.** Lines which are not perpendicular to each other are oblique.

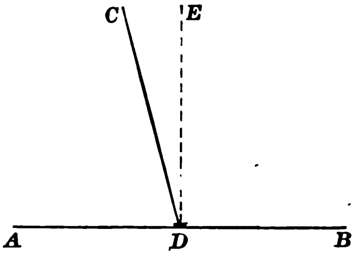
51. Two angles whose sum is equal to a right angle, or 90°, are called **complementary.** Each is the *complement* of the other. Thus, angles of 30° and 60° are complementary.

52. Two angles whose sum is equal to two right angles, or 180°, are called **supplementary.** Each is the *supplement* of the other. Angles of 45° and 135° are supplementary.

53. Complements of equal angles are equal, and supplements of equal angles are equal (proved by Axiom 4).

PROPOSITION I

54. Theorem. *If one straight line meets another, the adjacent angles are supplementary.*



Application. CD meets AB.
Prove CDA supplement of CDB

Construction. Draw $ED \perp AB$

Demonstration.

$CDA + CDB = EDA + EDB$ [sums are identical]

$EDA + EDB = 2$ rt. $\measuredangle$s [$ED \perp AB$]

$CDA + CDB = 2$ rt. $\measuredangle$s [subs. = for =]

or CDA is sup. of CDB

55. Scholium. The angles thus formed are called *supplementary adjacent* angles.

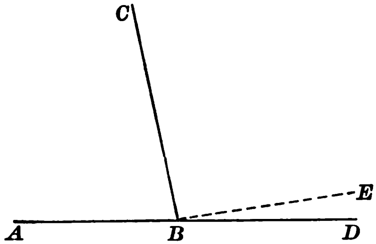
56. Corollary I. The sum of all the angles formed at a point on one side of a straight line is equal to two right angles.

57. Corollary II. The sum of all the angles around a point is equal to four right angles.

EXERCISE 1. If CDB is an angle of $105°$, how many degrees are there in the angle CDA?

PROPOSITION II

58. Theorem. *Conversely, if two adjacent angles are supplementary, their exterior sides form a straight line.*



Appl. *CBA* and *CBD* are supplementary adjacent angles.
Prove *ABD* a straight line

INDIRECT PROOF — COINCIDENCE

Cons. If *ABD* is *not* a straight line,
produce *AB*, making *ABE* a straight line

Dem. *CBD* = sup. of *CBA* [sup. adj. (hyp.)]
CBE = sup. of *CBA* [sup. adj. (cons.)]
CBD = *CBE* [sup. of same ∠]
BD coincides with *BE* [∠s = and *CB* common]
ABD = straight line [*ABE* = st. line]

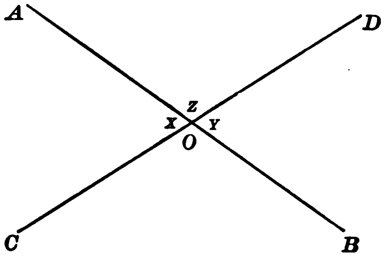
59. Cor. If the sum of any number of adjacent angles is equal to two right angles, the exterior sides of the outer angles form a straight line.

---

2. What is the complement of an angle of 30°? of 45°? of 77°? of 115°?

Proposition III

60. Theorem. *If two straight lines intersect, the vertical angles are equal.*



Appl. *AB* and *CD* intersect at *O*.
Prove $X = Y$

Dem. $X =$ sup. of Z [sup. adj. $\angle$s]
$Y =$ sup. of Z [sup. adj. $\angle$s]
$X = Y$ [sup. of same $\angle$]

EXERCISES

3. What is the supplement of an angle of 30°? of 45°? of 128°? of 315°?

4. How many degrees in the angle whose complement is one-fourth of its supplement?

5. How many degrees in an angle which is two-thirds of its complement?

6. How many degrees in the angle between the bisectors of two supplementary adjacent angles? two complementary adjacent angles?

7. What is true of the bisectors of two vertical angles?

8. Why can only one perpendicular be drawn to a line, at one point in the line.

9. Prove by coincidence that all right angles are equal.

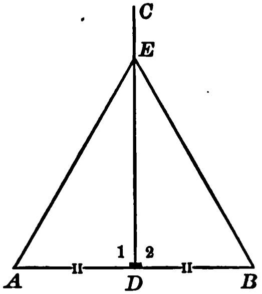
10. If in the above figure X is $\frac{4}{5}$ of a right angle, how many degrees are there in each angle?

PROPOSITION IV

61. Theorem. *If a perpendicular bisector is drawn to a line,*

1. *Any point in the perpendicular is equally distant from the extremities of the line.*

2. *Any point without the perpendicular is unequally distant from the extremities.*



Appl. $CD = \perp$ bisector of AB.

(Observe the graphic illustration of these two relations.)

Prove 1. $EA = EB$

2. $EA > EB$ (Fig. 2)

Case I

Cons. Fold EDB upon ED as an axis, until it lies on the opposite side of CD, still in the same plane

Dem. DB falls upon DA [1 = 2 (rt. ∡s)]

B " " A [$BD = AD$]

EB " " EA [one line bet. two pts.]

or E is equally distant from A and B

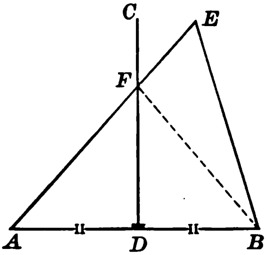


FIG. 2.

Case II

Cons. EA will cut CD at F.
Draw FB

Dem.

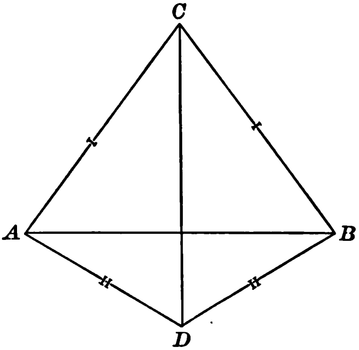
$EF + FB > EB$ [broken line > st. line]
$EF + FA > EB$ [subs. FA for FB (= by *Case* I)]
i.e. $EA > EB$

62. Cor. I. Lines drawn from a point in the perpendicular bisector of a line make equal angles with the perpendicular and with the line. Proved by the coincidence in *Case* I.

63. Cor. II. Every point which is equally distant from the extremities of a line lies in the perpendicular bisector of the line.

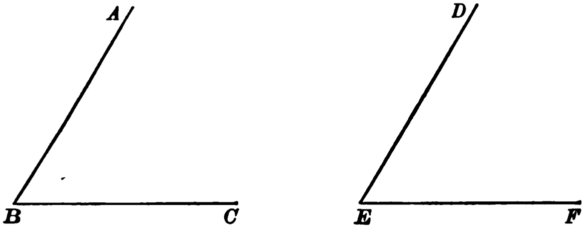
64. Cor. III. Two points equally distant from the extremities of a line determine the perpendicular bisector of the line.

Thus, if C is equally distant from A and B, and D is also equally distant from A and B, CD is a perpendicular bisector of AB.

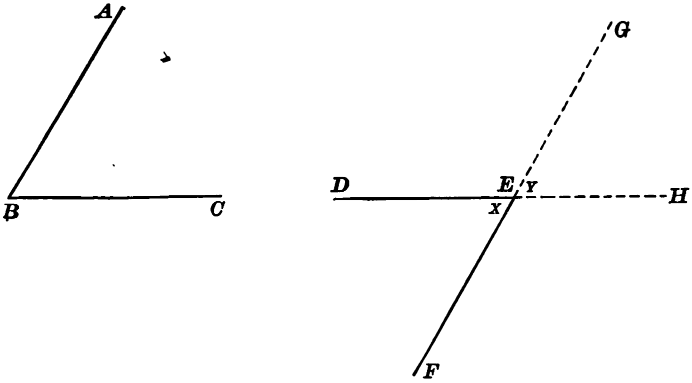


PROPOSITION V

65. Theorem. *Angles whose sides extend in the same or opposite directions are equal.*



Dem. $B = E$ [same *difference in direction* of sides]



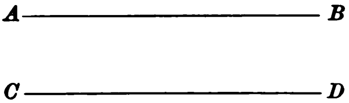
Appl. BA and EF, also BC and ED, have opposite directions.

Prove $B = X$

Cons. Produce FE and DE

Dem.
$X = Y$ [vert. ∠s]
$B = Y$ [sides have same direction]
$B = X$ [both $= Y$]

PARALLEL LINES



66. Parallel lines are such as have the same or opposite direction, as *AB* and *CD*.

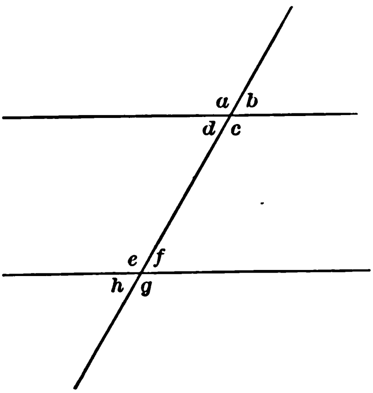
Parallel lines, however far produced, can never meet; and, conversely, lines in the same plane that never meet, however far produced, are parallel.

67. (*a*) Two lines parallel to a third line are parallel to each other.

(*b*) But one straight line can be drawn through a point parallel to a given straight line.

(*c*) Lines shown by demonstration to be parallel, and at the same time coincident at one point, coincide throughout and form one straight line.

68. When parallel lines are cut by another line (called a secant or transversal), as in the accompanying figure, the following names are given to the angles:



d and *f*, or *c* and *e*, are **alternate interior.**

a and *g*, or *b* and *h*, are **alternate exterior.**

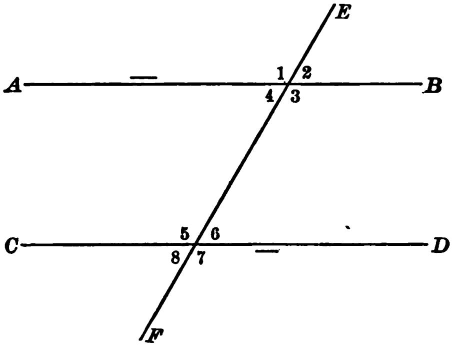
a and *e*, *d* and *h*, etc., are **corresponding.**

d and *e*, or *c* and *f*, are **consecutive,** on the same side of the secant.

PROPOSITION VI

69. Theorem. *If a secant cuts two parallel lines,*

1. *The alternate interior angles are equal.*
2. *The alternate exterior angles are equal.*
3. *The corresponding angles are equal.*
4. *The consecutive angles on the same side of the secant are supplementary.*



Appl. EF intersects ∥s AB and CD.

Prove $4 = 6$, etc.

Dem.

$$\text{alt. int. } \angle s \begin{cases} 4 = 6 & \text{[sides have opp. direction]} \\ 3 = 5 & [\text{`` \quad `` \quad `` \quad ``}] \end{cases}$$

$$\text{alt. ext. } \angle s \begin{cases} 1 = 7 & [\text{`` \quad `` \quad `` \quad ``}] \\ 2 = 8 & [\text{`` \quad `` \quad `` \quad ``}] \end{cases}$$

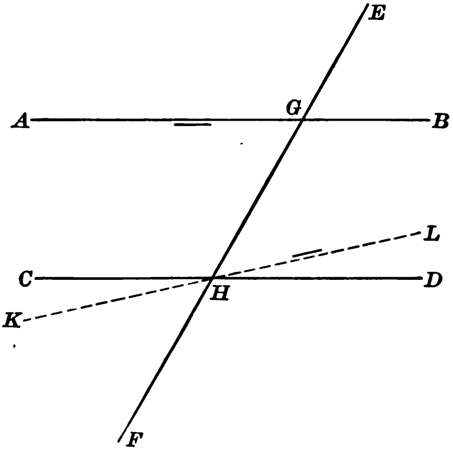
$$\text{corresp. } \angle s \begin{cases} 1 = 5 & \text{[sides have same direction]} \\ 2 = 6 & [\text{`` \quad `` \quad `` \quad ``}] \\ 3 = 7 & [\text{`` \quad `` \quad `` \quad ``}] \\ 4 = 8 & [\text{`` \quad `` \quad `` \quad ``}] \end{cases}$$

consec. ∠s
4 = sup. of 3 [sup. adj. ∠s]
4 = sup. of 5 [3 = 5 (alt. int. ∠s)]
3 = sup. of 6 [similarly]

PROPOSITION VII

70. Theorem. *Two lines intersected by a secant are parallel if,*

1. *the alternate interior angles are equal.*
2. *the alternate exterior angles are equal.*
3. *the corresponding angles are equal.*
4. *the consecutive angles on the same side of the secant are supplementary.*



Appl. AB is $\parallel$ to CD if alt. int. $\angle$s $AGH = GHD$, etc.

INDIRECT PROOF — COINCIDENCE

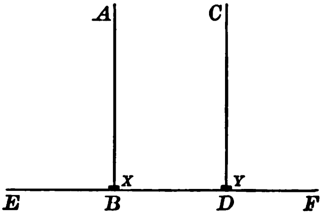
Cons. If CD is *not* $\parallel$ to AB,
draw KL, through H, $\parallel$ to AB

Dem. $GHL = AGH$ [alt. int. (cons.)]
$GHD = AGH$ [alt. int. (hyp.)]
$GHD = GHL$ [both $= AGH$]
HD coincides with HL [$\angle$s $=$ and GH common]
$CD \parallel AB$ [KL is $\parallel$ AB]

The last three cases are similarly proved.

PROPOSITION VIII

71. Theorem. *Two perpendiculars to the same line are parallel.*



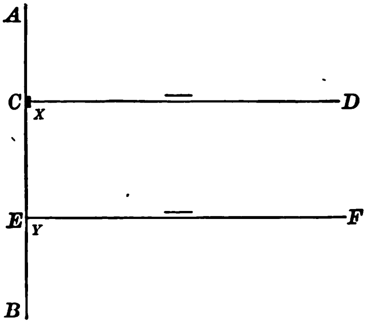
Appl. AB and $CD \perp EF$. Prove $AB \parallel CD$

Dem. $X = Y$ [rt. ∠s]

$AB \parallel CD$ [corresp. ∠s =]

PROPOSITION IX

72. Theorem. *A straight line perpendicular to one of two parallels is perpendicular to the other also.*



Appl. $CD \parallel EF$. $AB \perp CD$.

Prove $AB \perp EF$

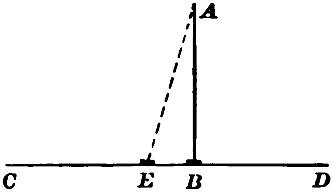
Dem. $X = Y$ [corresp. ∠s (∥s)]

$Y =$ rt. ∠ [$X =$ rt. ∠ (hyp.)]

or $AB \perp EF$

PROPOSITION X

73. Theorem. *Only one perpendicular can be drawn from a point to a straight line.*



Appl. *AB* is the only ⊥ that can be drawn from *A* to *CD*

INDIRECT PROOF — IMPOSSIBILITY

Cons. If *possible* draw a second ⊥, *AE*

Dem. *AB* ∥ *AE* [⊥s to same line]

This is impossible [∥s cannot meet]

∴ *AE* is not ⊥,

or, only one ⊥ can be drawn

EXERCISES

11. If the angle 2 (Theorem VI) is an angle of 72°, how many degrees in each of the other seven angles?

12. Two angles are supplementary, and the greater is five times the less. Find the number of degrees in each angle.

13. What angle is made by the hands of a watch at two o'clock? at seven o'clock?

14. What is the angle of rotation of the minute hand between twelve o'clock and four o'clock? of the hour hand?

15. In any given time, how does the angle of rotation of the minute hand compare with that of the second hand? with that of the second hand?

16. At seven and one-half minutes past five o'clock, what is the angle between the minute hand and the second hand? between the hour hand and the second hand?

Polygons — Definitions

74. A **polygon** is a plane figure bounded by straight lines.

The lines are called the **sides** of the polygon and their sum is called the **perimeter.**

A **diagonal** of a polygon is a line joining two vertices not consecutive.

A polygon is **convex** when no side, if produced, enters the space enclosed by the perimeter.

A polygon is **concave** when at least two sides, if produced, enter the enclosed space. Two such sides form an angle which is called **re-entrant.**

A polygon is convex unless otherwise indicated.

An **exterior** angle of a polygon is formed by one of its sides and an adjacent side produced.

The *angles* of a polygon are the angles between its sides, and the *vertices* of a polygon are the vertices of its angles.

An **equilateral** polygon is one whose sides are equal.

An **equiangular** polygon is one whose angles are equal.

Polygons are **mutually equilateral** when their sides are respectively equal, and **mutually equiangular** when their angles are respectively equal.

Homologous sides or angles, in mutually equiangular polygons, are those which are similarly situated.

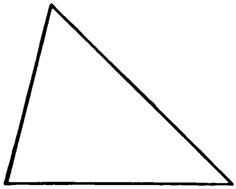
Polygons are equal when they may be so applied to each other as to exactly coincide.

A polygon of three sides is a triangle, or trigon; one of four sides, a quadrilateral, or tetragon; one of five, a pentagon; one of six, a hexagon; one of eight, an octagon; one of ten, a decagon; one of twelve, a dodecagon.

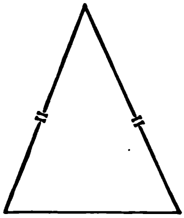
Triangles

75. A triangle is called **scalene** if no two sides are equal; **isosceles** if two sides are equal; and **equilateral** if its three sides are equal.

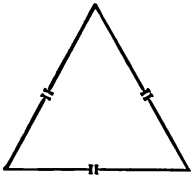
A triangle is called a **right triangle** if it has a right angle; **obtuse-angled triangle** if it has an obtuse angle; **acute-angled triangle** if it has three acute angles; and an **equiangular triangle** if its three angles are equal.



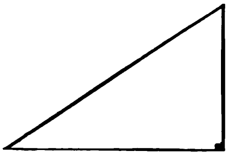
Scalene Triangle.



Isosceles Triangle.



Equilateral Triangle.



Right Triangle.

In a right triangle the side opposite the right angle is called the **hypotenuse,** and the other two sides the **legs.**

The **base** of a triangle, and likewise of any polygon, is the side upon which it is supposed to stand; but in an isosceles triangle the two equal sides are the legs, and the other side is the base.

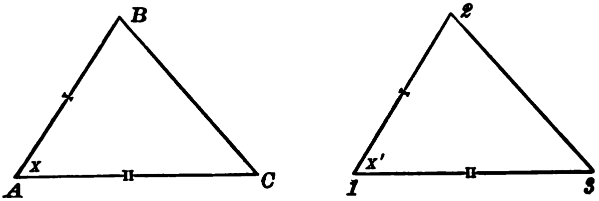
The **vertical angle** of a triangle is opposite the base, and its vertex is called the **vertex** of the triangle.

The **altitude** of a triangle is the perpendicular distance from the vertex to the base, or to the base produced.

A **median** is a line drawn from any vertex to the middle point of the opposite side.

PROPOSITION XI

76. Theorem. *Two triangles are equal if they have two sides and the included angle of one equal respectively to two sides and the included angle of the other.*



Appl.	I = I	
	II = II	
	$x = x'$	
	Prove $\triangle$s =	
Cons.	Place *AC* upon 1 3	[= by hyp.]
Dem.	*AB* falls upon 1 2	[$x = x'$]
	B " " 2	[I = I]
	BC " " 2 3	[one line bet. two pts.]
	$\triangle$s =	[coincide throughout]

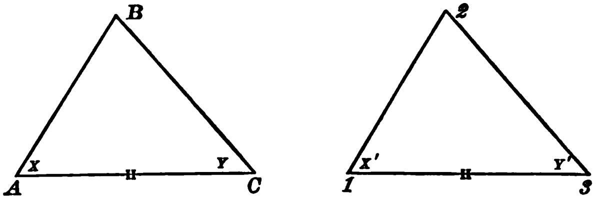
77. Cor. In the triangles here proved equal, the homologous sides *BC* and 2 3 are equal and the homologous angles *B* and 2 are equal, also *C* = 3 (proved by the coincidence).

78. In establishing *equality by means of coincidence*, the steps of proof should always conform to the following order:

1. A *common* line, as an axis, or a single line placed upon its equal.
2. *Coincidence of sides* adjacent to the first, because of *equality of angles*.
3. *Coincidence of points*, because of *equality of lines*, or because of *intersection* of coincident lines.

PROPOSITION XII

79. Theorem. ***Two triangles are equal if they have a side and two adjacent angles of one equal respectively to a side and two adjacent angles of the other.***



Appl. $\| = \|$

$X = X'$

$Y = Y'$

Prove $\triangle$s are =

Cons. Place AC upon 13 [= by hyp.]

Dem. AB falls upon 12 [$X = X'$]

CB " " 32 [$Y = Y'$]

B " " 2 [lies in 12 and 32]

$\triangle$s = [coincide throughout]

Sch. Note that from the equality of the triangles, the side $AB = 12$, that $CB = 32$, and that the angles B and 2 are equal.

EXERCISES

17. In a triangle ABC, $\angle A = \angle C$. Prove that a line parallel to AC makes equal angles with the sides BA and BC.

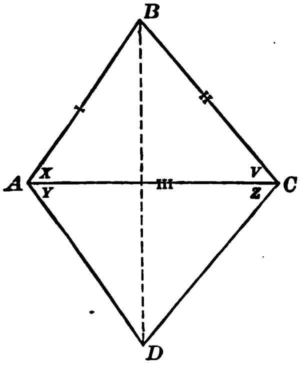
18. By means of angles prove that two parallels to a side of a triangle are parallel to each other.

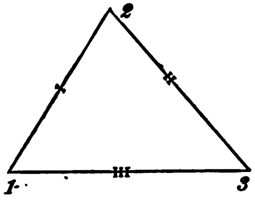
19. Why is any side of a triangle less than the sum of the other two sides?

20. Prove that one side of a triangle is greater than the difference of the other two.

PROPOSITION XIII

80. Theorem. *Two triangles are equal if they have the three sides of one equal respectively to the three sides of the other.*





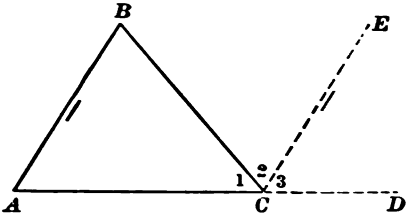
Appl. $AB = 12$ *or* I = I
$BC = 23$ " II = II
$AC = 13$ " III = III
Prove ◬ =

Cons. Place 123 in position ADC, 13 upon its =, AC.
Join BD

Dem.	$A =$ dist. from B and D	[$AB = 12 = AD$]
	$C =$ " " " " "	[$CB = 32 = CD$]
	$AC \perp$ bis. of BD	[two pts. = dist.]
	$X = Y$	[$\angle$s between oblique lines and $\perp$ bis. of line]
	$V = Z$	
	$\triangle ABC = \triangle ADC$	[s. and 2 adj. $\angle$s]
	$\triangle ABC = \triangle 123$	[$ADC = 123$]

PROPOSITION XIV

81. Theorem. *The sum of the angles of a triangle is equal to two right angles.*



Appl. Prove $\angle$s $A + B + 1 = 2$ rt. $\angle$s

Cons. Produce AC.
Draw $CE \parallel AB$

Dem.		
	$B = 2$	[alt. int. $\angle$s]
	$A = 3$	[corresp. $\angle$s]
	$A + B + 1 = 1 + 2 + 3$	[add $\angle$ 1 to each sum]
	$1 + 2 + 3 = 2$ rt. $\angle$s	[sum of $\angle$s at pt.]
	$A + B + 1 = 2$ rt. $\angle$s	[subs. = for =]

82. Cor. I. If one side of a triangle is produced,

1. the exterior angle is equal to the sum of the opposite interior angles;
2. the exterior angle is greater than either of the opposite interior angles.

83. Cor. II. Any angle of a triangle is the supplement of the sum of the other two angles.

84. Cor. III. If two triangles have two angles of one equal to two angles of the other, the third angles are equal.

85. Cor. IV. In any triangle there can be but one right angle or one obtuse angle.

86. Cor. V. In a right triangle the acute angles are complementary.

PROPOSITION XV

87. Theorem. *Two right triangles are equal if they have a leg and an acute angle of one equal to the homologous leg and acute angle of the other.*

Dem. By Cor. III or Cor. V of the preceding proposition, the other acute angles are equal and the triangles are then equal. [s. and two adj. $\angle$s].

Cor. By a similar process, it will be seen that two right triangles are equal if they have the hypotenuse and an acute angle of one equal to the hypotenuse and the homologous acute angle of the other.

EXERCISES

21. If in figure on page 28 the angle $B = 75°$ and the angle $1 = 38°$, how many degrees in the angles A, 2, and 3?

22. If the vertical angle of an isosceles triangle contains 36°, how many degrees are there in each base angle?

23. If a base angle of an isosceles triangle contains 28°, how many degrees are there in each of the remaining angles?

24. How many degrees are there in each angle of an equiangular triangle?

25. How many degrees are there in each acute angle of a right isosceles triangle?

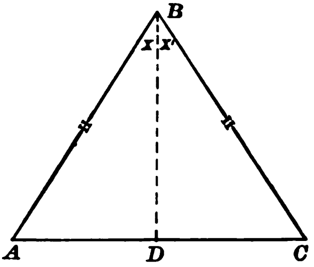
26. How many degrees in each angle of an isosceles triangle if the vertical angle is equal to the sum of the base angles? twice the sum? one-half the sum? In the first and last cases, what kind of triangles are formed?

27. The sum of the lines drawn from a point within a triangle to three vertices is less than the perimeter and also greater than the semi-perimeter.

28. Two isosceles triangles are equal if they have any side and any angle of one equal to the homologous side and angle of the other.

PROPOSITION XVI

88. Theorem. ***In an isosceles triangle the angles opposite the legs are equal.***



Appl. $\| = \|$

Prove $A = C$

Cons. Draw BD bisecting $\angle B$

Dem. $\triangle ABD = \triangle CBD$

$$\left[\begin{array}{c}\text{2 s. and inc. }\angle\\ \hline BD\text{ com.}\\ \| = \|\\ x = x'\end{array}\right]$$

$\therefore\ A = C$

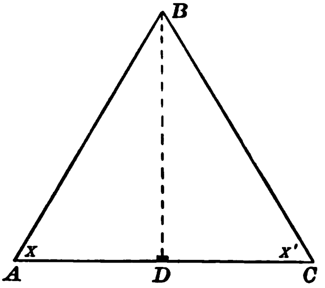
89. Cor. I. In the equal $\triangle$s, $AD = DC$, the $\angle$s at D are $=$, and therefore in an isosceles triangle, a line which satisfies any two of the following conditions satisfies all:

1. Drawn through the vertex.
2. Drawn bisecting the vertical angle.
3. Drawn perpendicular to the base.
4. Drawn bisecting the base.

90. Cor. II. An equilateral triangle is also equiangular.

PROPOSITION XVII

91. Theorem. *If two angles of a triangle are equal, the sides opposite are equal.*



Appl. $X = X'$

Prove $BA = BC$

Cons. Draw $BD \perp AC$

Dem. $\triangle ABD = \triangle CBD$ [rt. △s, leg and acute ∠ / BD com / $X = X'$]

$\therefore BA = BC$

92. Cor. An equiangular triangle is also equilateral.

EXERCISES

29. If the legs of an isosceles triangle are produced below the base, the exterior angles formed are equal.

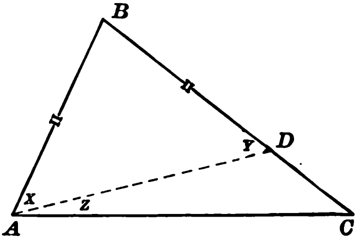
30. The ∠s A and C of a triangle ABC are respectively 42° and 92°; how many degrees are there in the exterior angle formed at the vertex B?

31. The ∠ A of a triangle ABC is five times the ∠ C, and the ∠ B is three times the ∠ C; how many degrees are there in each angle?

32. The exterior angle at the vertex of an isosceles triangle is 114°. How many degrees are there in each exterior angle at the extremities of the base?

PROPOSITION XVIII

93. Theorem. *If two sides of a triangle are unequal, the angles opposite are unequal, the greater angle lying opposite the greater side.*



Appl. $BC > BA$.

Prove $A > C$

Cons. Make $BD = BA$.

Draw AD

Dem.

$X = Y$	[base ∠s of isos. △]
$Y > C$	[ext. ∠ of △ ($Y = C + Z$)]
1. $X > C$	[subs. X for Y]
2. $A > X$	[whole > part]
$A > C$	[from 1 and 2]

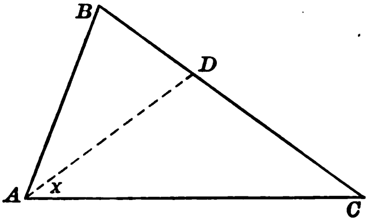
EXERCISES

33. The exterior angles at two vertices of a triangle are respectively 132° and 75°. How many degrees are there in each interior angle, and how many in the exterior angle at the third vertex?

34. If the perpendicular from the vertex to the base of a triangle also bisects the base, the triangle is isosceles.

PROPOSITION XIX

94. Theorem. *If two angles of a triangle are unequal, the sides opposite are unequal, the greater side lying opposite the greater angle.*



Appl. $A > C$.

Prove $BC > BA$

Cons. Draw AD making $x = C$

Dem. $DA = DC$ [isos. △ ($x = C$)]

$BD + DA > BA$ [broken l. > st. l.]

$BD + DC > BA$ [subs. DC for DA]

i.e. $BC > BA$

EXERCISES

35. The vertical angle of an isosceles triangle is three-halves the sum of the equal base angles. How many degrees are there in each angle?

36. In the figure of Theorem XVIII., if the $\angle B = 65°$ and $Z = 32°$, how many degrees are there in the $\angle BAD$? in the $\angle C$?

37. Bisectors of the base angles of an isosceles triangle are equal.

38. Perpendiculars from the extremities of the base of an isosceles triangle to the legs are equal.

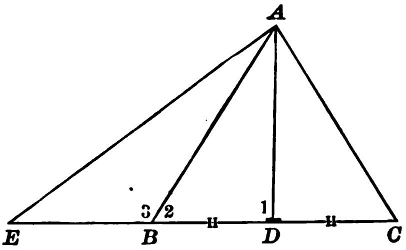
39. Medians to the legs of an isosceles triangle are equal.

40. The perpendicular to the base of an isosceles triangle at an extremity of the base and meeting the other leg produced, forms a second isosceles triangle exterior to the first.

PROPOSITION XX

95. Theorem. *If from a point without a line a perpendicular and oblique lines are drawn to this line,*

1. *The perpendicular is shorter than any oblique line.*
2. *Two oblique lines cutting off equal distances from the foot of the perpendicular are equal.*
3. *Of two oblique lines the more remote is the greater.*



Appl. Prove 1. $AD < AB$,

2. $AB = AC$,

3. $AE > AB$

Dem.

$2 =$ acute $\angle$ [$1 =$ only possible rt. $\angle$]

1. $AD < AB$ [AB opposite greater $\angle$]

2. $AB = AC$ [$AD = \perp$ bis. of BC (hyp.)]

$3 =$ obtuse $\angle$ [sup. of acute $\angle$ 2]

$E =$ acute $\angle$ [$3 =$ only possible obtuse $\angle$]

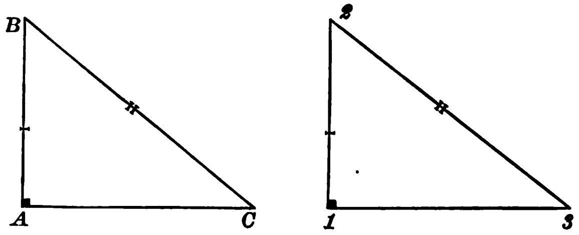
3. $AE > AB$ [opp. greater $\angle$]

96. Cor. I. Converse to *Case* II. Equal oblique lines cut off equal distances from the foot of the perpendicular.

97. Cor. II. Converse to *Case* III. The greater oblique line cuts off the greater distance from the foot of the perpendicular.

Proposition XXI

98. Theorem. *Two right triangles are equal if they have the hypotenuse and a leg of one equal to the hypotenuse and the homologous leg of the 'other.*



Appl. $| = |$

$\| = \|$

Prove $\triangle$s =

Cons. Place AB upon its =, 12

Dem. AC falls upon 13 [$A = 1$ (rt. $\angle$s)]

$AC = 13$ [= obl. l. cut off = dist. from foot of $\perp$]

$\triangle$s = [3 s. = 3 s.]

SUMMARY

The **Equality of Triangles** is established by preceding demonstrations, which may be summarized as follows:

Scalene Triangles. (1) 2 s. and inc. $\angle$

(2) s. and two adj. $\angle$s

(3) 3 s.

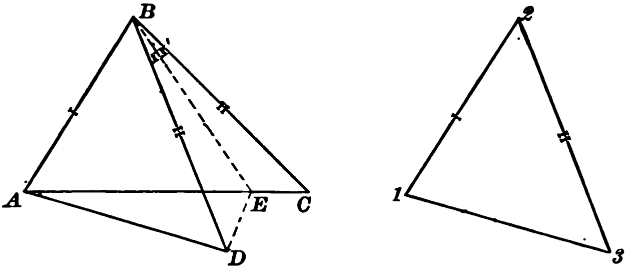
Right Triangles. (1) hyp. and s.

(2) hyp. and acute $\angle$

(3) s. and acute $\angle$

Proposition XXII

99. Theorem. *If two triangles have two sides of one respectively equal to two sides of the other, but the included angles unequal, the third sides are respectively unequal.*



Appl. | = |

|| = ||

$B > 2$

Prove $AC > 13$

Cons. Place $\triangle\ 123$ in position ABD,

12 upon its =, AB, and 3 at D.

Draw BE bisecting $\angle\ DBC$ and join DE

Dem. $\triangle BED = \triangle BEC$ [2 s. and inc. ∠ / BE com. / || = || / $x = x'$]

$ED = EC$

$AE + ED > AD$ [broken line > st. l.]

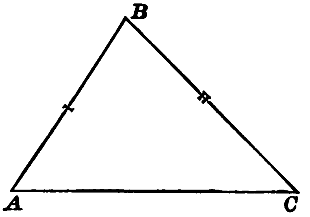
AC or $AE + EC > AD$ [subs. EC for ED]

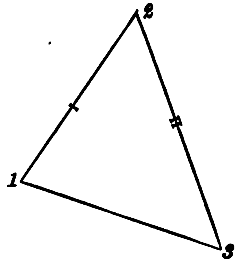
$AC > 13$ [subs. 13 for AD]

Sch. By the word "respectively" in the theorem is meant, that the third side is greater in the triangle in which the angle between the first sides is greater.

PROPOSITION XXIII

100. Theorem. *If two triangles have two sides of one respectively equal to two sides of the other, but the third sides unequal, the angles opposite these third sides are respectively unequal.*





Appl. $\mid \; = \; \mid$

$\mid\mid \; = \; \mid\mid$

$AC > 13$

Prove $B > 2$

INDIRECT PROOF — EXHAUSTION

Dem. *Either* 1. $B = 2$

or 2. $B < 2$

or 3. $B > 2$ [all possible suppositions]

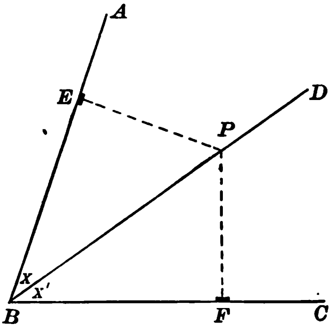
1. B does not $= 2$ [If $B = 2$, $AC = 13$ (△s =, 2 s. and inc. ∠). (contradicts hyp.)]

2. B is not < 2 [If $B < 2$, $AC < 13$ (preceding theorem). (contradicts hyp.)]

3. $\therefore B > 2$ [Two suppositions exhausted, ∴ third is proved]

PROPOSITION XXIV

101. Theorem. *Every point in the bisector of an angle is equally distant from the sides.*



Appl. DB bisects ABC. Prove $P =$ dist. from sides

Cons. Draw PE and PF ⊥ to sides of ∠

Dem. rt. $\triangle$s = [hyp. and acute ∠ — BP com. — $X = X'$]

∴ $PE = PF$

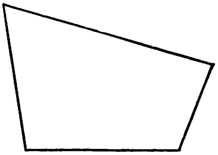
$P =$ dist. from sides [⊥s =]

QUADRILATERALS — DEFINITIONS

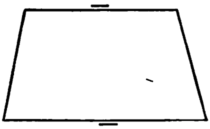
102. A **trapezium** is a quadrilateral which has no two sides parallel.

A **trapezoid** is a quadrilateral which has two sides, and only two, parallel.

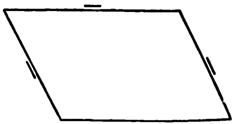
A **parallelogram** is a quadrilateral which has its opposite sides parallel.



TRAPEZIUM.



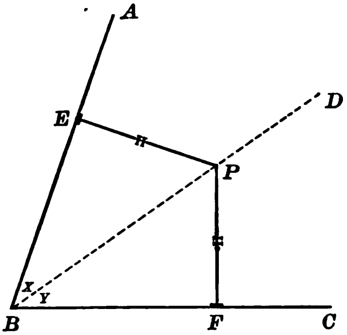
TRAPEZOID.



PARALLELOGRAM.

PROPOSITION XXV

103. Theorem. *Conversely, any point equally distant from the sides of an angle is in the bisector.*



Appl. $PE = PF$. Prove P is in bisector of $\angle$

Cons. Through P draw DB

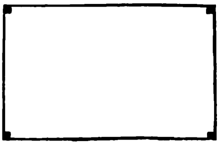
Dem. rt. $\triangle$s $=$ $\left[\begin{array}{c} \text{hyp. and s.} \\ \hline BP \text{ com} \\ PE = PF \end{array}\right]$

$\therefore X = Y$

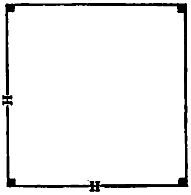
i.e. DB is a bisector, *or* P lies in the bisector

QUADRILATERALS — DEFINITIONS

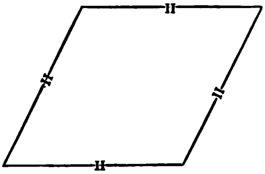
104. A **rectangle** is a right-angled parallelogram.
A **square** is an equilateral rectangle.
A **rhomboid** is an oblique-angled parallelogram.
A **rhombus** is an equilateral rhomboid.



RECTANGLE.



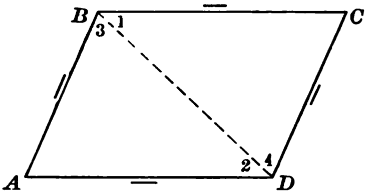
SQUARE.



RHOMBUS.

PROPOSITION XXVI

105. Theorem. *In a parallelogram the opposite sides are equal and the opposite angles are equal.*



Appl. $ABCD = \square$

Prove 1. $BC = AD$, and $BA = CD$

2. $A = C$, and $B = D$

Cons. Draw BD

Dem.	$1 = 2$	[alt. int. ∠s (∥s)]
	$3 = 4$	[alt. int. ∠s (∥s)]
	△s =	[s. and 2 adj. ∠s]
	∴ 1. $BC = AD$	homol. parts of = △s
	$BA = CD$	
	2. $A = C$	
or	2. $A = C$	sides ∥ each to each, extending in opp. direction
	$B = D$	

106. Cor. The consecutive angles of a parallelogram are supplementary.

TRAPEZOIDS — DEFINITIONS

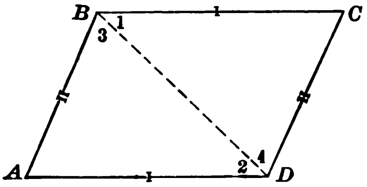
107. The **bases** of a trapezoid are its parallel sides.

The non-parallel sides are its legs, and if the legs are equal the trapezoid is **isosceles.**

The **median** of a trapezoid joins the middle points of the legs.

Proposition XXVII

108. Theorem. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Appl.	$\mid = \mid$, $\Vert = \Vert$.	Prove $ABCD = $ ▱
Cons.	Draw BD	
Dem.	$\triangle$s $=$	[3 s. = 3 s.]
	$\therefore 1 = 2$	[homol. parts]
	$3 = 4$	
	$BC \parallel AD$	[1 = 2, alt. int. $\angle$s]
	$BA \parallel CD$	[3 = 4, alt. int. $\angle$s]
	$\therefore ABCD = $ ▱	

EXERCISES

41. How many degrees are there in each angle of a parallelogram, if one angle is 65° ? 144° ?

42. How many degrees, if the sum of two opposite angles is 162° ?

43. How many degrees, if the difference of two adjacent angles is 32° ?

44. The diagonals of a rectangle are equal.

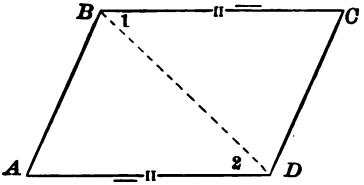
45. If the diagonals of a parallelogram are equal, the figure is a rectangle.

46. Prove by coincidence that two parallelograms are equal if they have two sides and the included angle of one equal respectively to two sides and the included angle of the other.

47. If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are equal.

PROPOSITION XXVIII

109. Theorem. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Appl. $BC =$ and $\parallel AD$. Prove $ABCD = ▱$

Cons. Draw BD

Dem. $1 = 2$ [alt. int. ∠s ($\parallel$s)]

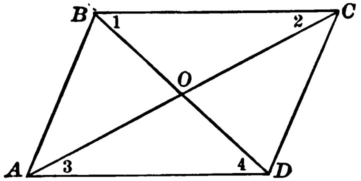
△s $=$ [2 s. and inc. ∠]

$CD = BA$ [homol. parts]

$ABCD = ▱$ [opp. sides $=$]

PROPOSITION XXIX

110. Theorem. *The diagonals of a parallelogram bisect each other.*



Appl. Prove $BO = DO$, and $AO = CO$

Dem. $1 = 4$ [alt. int. ∠s ($\parallel$s)]

$2 = 3$ [alt. int. ∠s ($\parallel$s)]

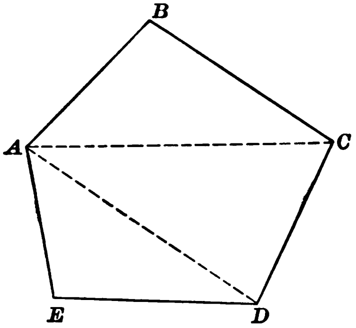
$\triangle BCO = \triangle ADO$ [s. and 2 adj. ∠s]

$\therefore BO = DO$

$AO = CO$

PROPOSITION XXX

111. Theorem. *The sum of the angles of a polygon is equal to twice as many right angles as the figure has sides, less four right angles.*



Appl. Prove $A + B + C$, etc. $= (2n - 4)$ rt. $\angle$s

Cons. Draw diagonals from one vertex.
Let $n =$ the number of sides

Dem. $n - 2 =$ no. of $\triangle$s [In pairing $\triangle$s with sides, AB and AE are in excess]

sum $\angle$s of ⬠ $= (n - 2)(2$ rt. $\angle$s$)$ [sum $\angle$s of ⬠ = sum $\angle$s of all $\triangle$s. sum $\angle$s of each $\triangle = 2$ rt. $\angle$s]

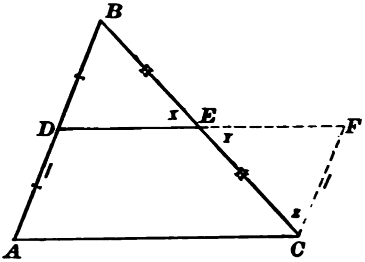
$= (2n - 4)$ rt. $\angle$s [algebraic expansion]

112. Cor. The sum of the exterior angles of any polygon is equal to four right angles.

[The sum of ext. and int. $\angle$s at each vertex $= 2$ rt. $\angle$s, at n vertices $= 2n$ rt. $\angle$s. $\therefore$ the sum of ext. $\angle$s $= 2n - (2n - 4) = 4$.]

PROPOSITION XXXI

113. Theorem. *The line joining the middle points of two sides of a triangle is parallel to the third side and equal to one-half of it.*



Appl. | = |

|| = ||

Prove $DE \parallel AC$ and $= \frac{1}{2} AC$

Cons. Draw $CF \parallel AB$,

meeting DE produced at F

Dem. $\triangle BDE = \triangle EFC$ [s. and 2 adj. ∠s; || = ||; $x = y$ (vert. ∠s); $z = B$ (alt. int. ∠s)]

$DE = EF$

$CF = BD$

$AD = CF$ [both $= BD$]

$ADFC = ▱$ [2 opp. s. = and ∥]

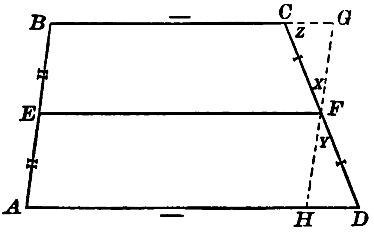
$DF \parallel AC$ and $= AC$ [opp. s. of ▱]

$DE \parallel AC$ and $= \frac{1}{2} AC$ [$DE = EF = \frac{1}{2} DF$]

114. Cor. A line drawn from the middle point of one side of a triangle parallel to a second side bisects the third side, [proved by noting its coincidence with a line drawn as in the theorem (two ∥s to same line through one point, impossible)].

Proposition XXXII

115. Theorem. *The line joining the middle points of the legs of a trapezoid is parallel to the bases and equal to one-half their sum.*



Appl. $| = |$, $|| = ||$

Prove $EF \parallel BC$ and AD, also $EF = \frac{1}{2}(BC + AD)$

Cons. Through F draw $GH \parallel BA$,
meeting BC produced at G

Dem. $\triangle CFG = \triangle DFH$ [s. and 2 adj. $\angle$s; $| = |$; $X = Y$ (vert.); $Z = D$ (alt. int. $\angle$s)]

$GF = FH$

$CG = HD$

$BA = GH$ [opp. s. of ▱]

$BE = GF$ [halves of $=$s]

$BGFE = $ ▱ [2 opp. s. $=$ and $\parallel$]

$EF \parallel BG$ and $= BG$ [opp. s. of ▱]

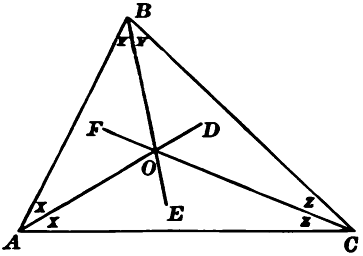
$EF \parallel AH$ and $= AH$ [similarly]

$$2\,EF = BG + AH$$
$$= (BC + CG) + (AD - HD)$$
$$= BC + AD \quad [CG = HD\,(= \triangle\text{s})]$$
$$EF = \tfrac{1}{2}(BC + AD)$$

116. Cor. A line drawn from the middle point of one leg of a trapezoid, parallel to the bases, bisects the other leg.

PROPOSITION XXXIII

117. Theorem. *The three bisectors of the angles of a triangle meet at a common point.*



Appl. $X = X$, $Y = Y$, $Z = Z$.

Prove AD, BE, and CF are concurrent

Cons. Let AD and CF intersect at O

Dem. 1. O is = dist. from AC and AB [in bisector of $\angle A$]

2. O " = " " AC " BC [" " " $\angle C$]

O " = " " AB " BC [from 1 and 2]

O is in bisector of $\angle B$ [= dist. from AB and BC]

i.e. BE passes through O

or the three bisectors are concurrent

118. Cor. The point O is equally distant from the three sides of the triangle.

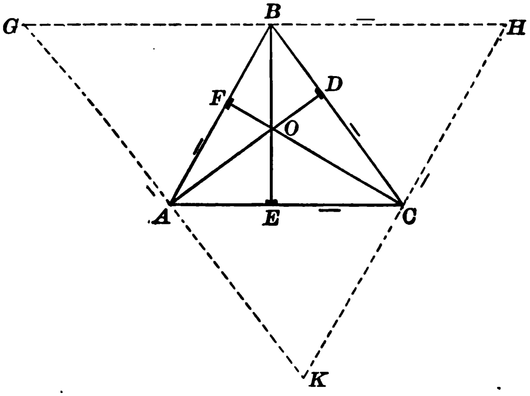
PROPOSITION XXXIV

119. Theorem. *The perpendicular bisectors of the sides of a triangle meet at a common point.*

HINT. Prove by a process similar to that of the preceding proposition.

PROPOSITION XXXV

120. Theorem. *The perpendiculars from the vertices of a triangle to the opposite sides meet at a common point.*



Appl. Prove $\perp$s AD, BE, and CF are concurrent

Cons. Through vertices A, B, and C draw GK, GH, and HK $\parallel$ to the opposite sides of the triangle

Dem. $EB \perp GH$ [$\perp$ to AC ($AC \parallel GH$)]

$GB = AC$ [opp. s. of ▱]

$BH = AC$ [opp. s. of ▱]

$GB = BH$ [both $= AC$]

i.e. $BE = \perp$ bis. of GH

$AD = \perp$ " " GK

$CF = \perp$ " " HK [similarly]

AD, BE, and CF are concurrent [$\perp$ bisectors of s. of $\triangle$]

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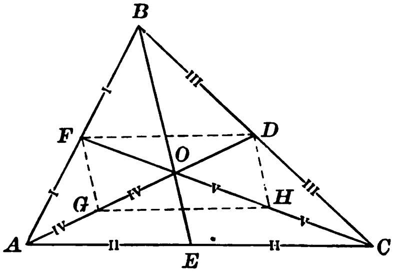
48. How many right angles in the sum of the angles of a hexagon, octagon, etc. ?

PROPOSITION XXXVI

121. Theorem. *The medians of a triangle meet at a common point.*



Appl. I = I
II = II
III = III
Prove AD, BE, and CF are concurrent

Cons. Bisect AO and CO.
Connect FD, DH, HG, GF

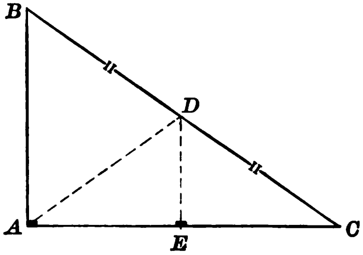
Dem.

$FD \parallel AC$ and $= \frac{1}{2}AC$	[joins mid. pts. of 2 s. of $\triangle ABC$]
$GH \parallel AC$ and $= \frac{1}{2}AC$	[" " " " 2 s. " $\triangle AOC$]
$FD =$ and $\parallel GH$	[$\parallel$ same line AC and $= \frac{1}{2}$ of it]
$FDHG = ▱$	[2 opp. s. = and $\parallel$]
$GO = OD$	[diags. of ▱ bisect each other]
i.e. AD is trisected	[$GO = AG$ (cons.) and $= OD$]
$\therefore AO = \frac{2}{3}AD$	
BE intersects AD at O	[similarly (*i.e.* at a point $\frac{2}{3}$ dist. from A to D)]
$\therefore AD$, BE, and CF are concurrent	

122. Sch. The centre of gravity of a triangular board is determined by the intersection of two medians.

Proposition XXXVII

123. Theorem. *The middle point of the hypotenuse of a right triangle is equally distant from the three vertices.*



Appl. || = ||

Prove D is equally distant from A and C, and hence from A, B, and C.

Cons. Draw DA.

Draw $DE \perp AC$

Dem.	$DE \parallel BA$	[two $\perp^s$ to same line]
	DE bisects AC	[bisects BC and $\parallel BA$]
	$D =$ dist. from A and C	[in $\perp$ bis. of AC]
	$\therefore D$ is equally dist. from A, B, and C	

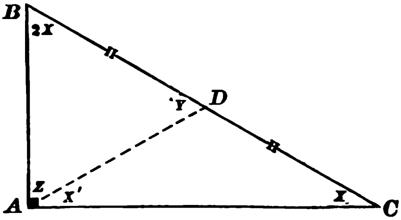
124. Cor. A median from the vertex of the right angle to the hypotenuse divides a right triangle into two isosceles triangles.

49. How many degrees in each angle of an equiangular pentagon? hexagon? octagon? decagon? dodecagon?

50. How many sides has the polygon, the sum of whose interior angles is twice the sum of its exterior angles? four times? one-half?

PROPOSITION XXXVIII

125. Theorem. *If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.*



Appl. $B = 2\,C$, *i.e.* $2\,X$

Prove $BC = 2\,BA$

Cons. Draw median AD

Dem. $X' = X$ [isos. △ (AD median to hypot.)]

$Z = 2\,X$ [BDA = isos. △]

$Y = 2\,X$ [sum of X and X' (ext. ∠ of △)]

BDA = equilat. △ [equi. ∠ △]

$BD = BA$

$\therefore BC = 2\,BA$ *or*

Dem. $X + 2\,X = 90°$ [acute ∠s of. rt △]

$X = 30°$ [solving]

$2\,X = 60°$

$Z = 60°$ [comp. of X']

$Y = 60°$ [sup. of $(60° + 60°)$]

BDA = equilat. △, etc.

126. Sch. A triangle of this shape is used extensively in mechanical draughting and in the earliest principles of trigonometry.

LOCI — DEFINITIONS

127. A **geometric locus** is the position of all the points which satisfy a given condition. A locus is a line or a combination of lines and points. Frequently two perpendiculars or two parallels form a locus, according to the conditions.

Thus, as proved by Theorem IV, the locus of points equally distant from the extremities of a line, is the perpendicular bisector of the line. And again (see Theorems XXIV and XXV), the locus of points equally distant from the sides of an angle is the bisector of the angle.

A locus may also be defined as the line traced by a point which moves according to a given law. The locus includes all the points which satisfy the condition, and also every point upon the locus satisfies the condition.

128. A locus of the points which satisfy a given condition is best determined by locating as nearly as possible a single point satisfying the condition, and then several points in succession, all satisfying the condition and occupying as many different positions as possible. The line indicated by these points is the required locus.

EXERCISES

51. What is the locus of points at a given distance from a given straight line?

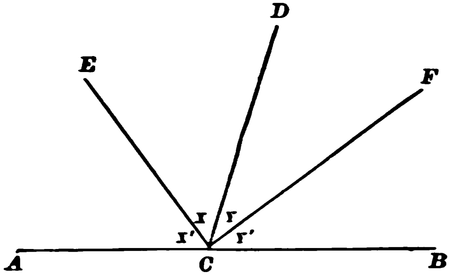
52. What is the locus of points equally distant from a pair of intersecting straight lines? What relation exists between the two parts of the locus?

53. What is the locus of points equally distant from a pair of parallel straight lines?

54. What is the locus of the middle points of lines drawn from a given point to an indefinite line?

55. If a series of equal parallels, extending in the same direction, is drawn from points in the perimeter of a polygon, what is the locus of the free extremities of these parallels?

PROPOSITIONS OF INTERPRETATION



Ex. 56. $ACB =$ st. line. $ECF =$ what?

129. Interpret the figure, and prove the theorem implied in the question concerning ECF.

[Interpretation: DC meets a st. line, forming (Theo. 1) suppl. adj. ∠s; $X = X'$ and $Y = Y'$; *i.e.* the ∠s DCA and DCB are bisected.

The theorem to be proved evidently is:

The bisectors of supplementary adjacent angles are perpendicular to each other.]

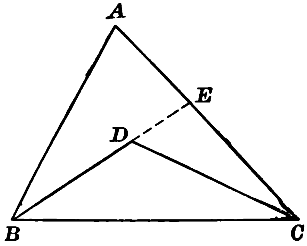
OUTLINE OF THE PROOF

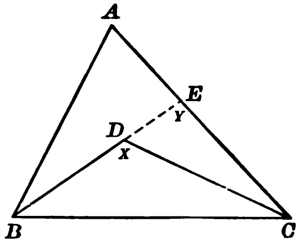
$X' + X + Y + Y' = 2$ rt. ∠s [sum of ∠s at pt.]
i.e. $2X + 2Y = 2$ rt. ∠s
$X + Y =$ rt. ∠ [dividing by 2]
i.e. EC is $\perp CF$

The interpretation of the figure suggests the several details which lead to and constitute the hypothesis of the theorem. Upon these details the proof is based.

There are frequently several theorems that are suggested by the graphic figure. As many deductions as possible should be made from the hypothesis shown in the figure.

INTERPRET AND PROVE





Ex. 57. Prove $BA + AC > BD + DC$. Prove $X > A$

Interpretation:

$ABC = any\ \triangle$ [There are no marks showing it to be right, isosceles, or equilateral]

D is *any* point

The two theorems to be proved must be:

If lines are drawn from any point within a triangle to the extremities of a side,

1. *Their sum is less than the sum of the other two sides.*

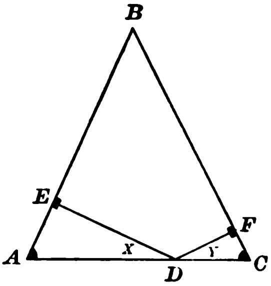
2. *The angle included by them is greater than the angle included by the other two sides.*

OUTLINE OF THE PROOF

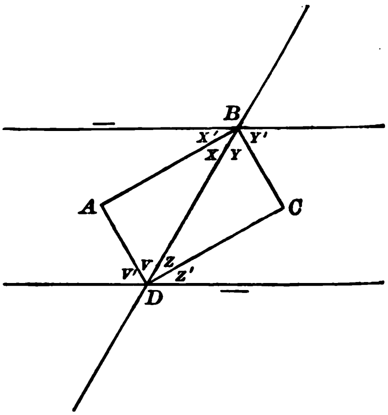
Produce BD to E

1.	$BA + AE > BE$	[?] [reason to be supplied]
(a)	$BA + AC > BE + EC$	[Add EC to each member of inequality]
	$DE + EC > DC$	
(b)	$BE + EC > BD + DC$	[what is added to each member of inequality ?]
	$BA + AC > BD + DC$	[combine (*a*) and (*b*)]
2.	$X > Y$	[?]
	$Y > A$	[?]
	$X > A$	

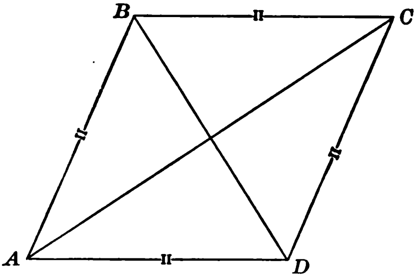
INTERPRET AND PROVE



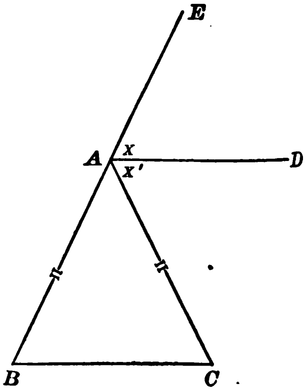
Ex. 58. Prove $X = Y$



Ex. 59. $ABCD =$ what?

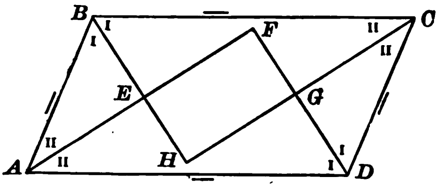


Ex. 60. Prove AC and BD bisect each other at rt. $\angle$s.
Prove other relations between the different angles and the triangles.



Ex. 61. (*a*) Prove $AD \parallel BC$.
(*b*) State and prove two converse theorems.

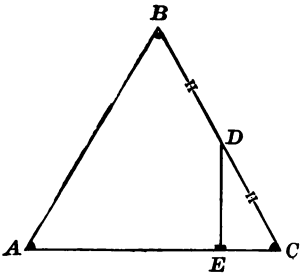
INTERPRET AND PROVE



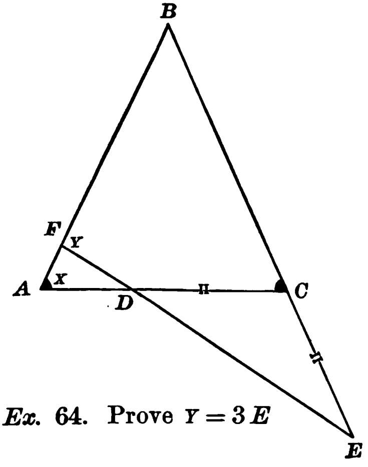
Ex. 62. 1. *EFGH* = what?

2. What is true of the bisectors of the angles of a trapezoid?

3. Of a rectangle?

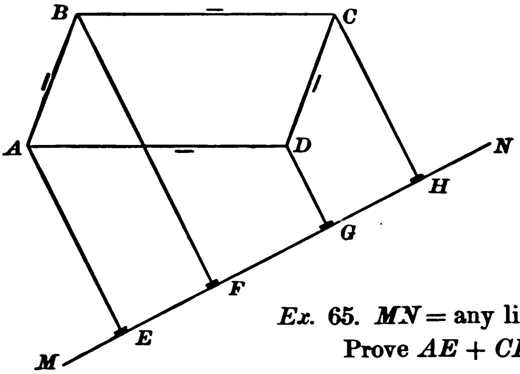


Ex. 63. Prove $EC = \frac{1}{4}AC$



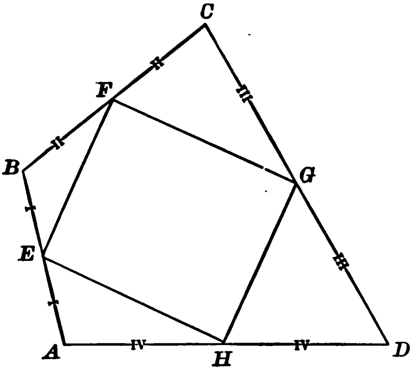
Ex. 64. Prove $Y = 3E$

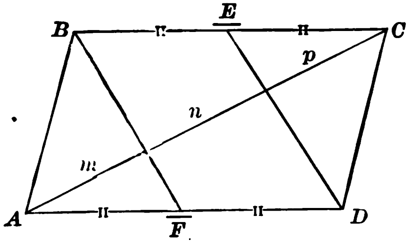
INTERPRET AND PROVE



Ex. 65. $MN =$ any line.
Prove $AE + CH = BF + DG$

Ex. 66. $EFGH =$ what?





Ex. 67. Prove $m = n = p$

EXERCISES

68. The base of an isosceles triangle and the bisectors of the base angles form an isosceles triangle.

69. Two isosceles triangles are equal if the base and the vertical angle of one equals the base and the vertical angle of the other.

70. An exterior angle at the base of an isosceles triangle is equal to the angle between the bisectors of the interior base angles.

71. If a secant between two parallels is bisected, any other secant drawn through its middle point, and included between the parallels, is bisected.

72. The perpendiculars from two opposite vertices of a parallelogram to the diagonal which joins the other vertices are equal.

73. Points in the legs of an isosceles triangle which are equally distant from the vertex are also equally distant from the base.

74. Parallels to the legs of an isosceles triangle from any point in the base, complete a parallelogram whose perimeter equals the sum of the legs of the triangle.

75. One side of an angle, its bisector, and a parallel to the other side, form an isosceles triangle.

76. ABC and DBC are two triangles on the same base BC and on the same side of it, such that $AB = DC$, $AC = DB$; and AC intersects DB at O. Prove that BOC is isosceles.

77. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

78. A re-entrant angle of a quadrilateral, exterior to the figure, is equal to the sum of the three interior angles at the other vertices.

79. The perpendicular to the leg of an isosceles triangle from the opposite vertex makes with the base an angle equal to one-half the angle at the vertex of the triangle.

80. D is any point in AB, a leg of an isosceles triangle. DE is drawn perpendicular to the base BC, meeting CA produced at F. Prove that the triangle ADF is isosceles.

81. The diagonals and also the base angles of an isosceles trapezoid are equal.

82. If the bases of a trapezoid are perpendicular to one of the legs, the lines drawn from the extremities of this leg to the middle point of the other leg are equal.

83. ABC is an isosceles triangle and BC its base. The angle A is twice the sum of B and C. CD is drawn perpendicular to the base, meeting BA produced at D. Prove ACD equilateral.

84. The perpendiculars from two vertices of a triangle to the median, produced if necessary, drawn from the other vertex of the triangle, are equal.

85. D is the middle point of BC, in a triangle ABC, and E is the middle point of AD. Prove that BE produced trisects AC.

86. D, E, and F are points on the sides AB, BC, and CA of an equilateral triangle ABC, equally distant from A, B, and C, respectively. Prove that DEF is an equilateral triangle.

87. E, F, G, and H are points on the four sides of a square $ABCD$, such that $AE = BF = CG = DH$. Prove that $EFGH$ is first a rhombus and then a square.

88. $ABCD$ is a square. On the diagonal BD, a distance BE is made equal to a side of the square, and EF is drawn perpendicular to the diagonal, meeting AD at F. Prove that $AF = EF = ED$.

89. D is any point on the side AC of a triangle ABC, and E, F, G, and H are the middle points of AD, CD, BC, and AB, respectively. Prove that $EFGH$ is a parallelogram.

90. If AD is drawn from A, the vertex of the right angle of a right triangle to the hypotenuse, so as to make the angle $DAB = \angle B$, it bisects the hypotenuse.

91. AB is the hypotenuse of a right triangle. The bisectors of the exterior angles at A and B meet at D, and perpendiculars DE and DF are drawn respectively to CA and CB produced. Prove that $CEDF$ is a square.

92. CD bisects the angle C of the triangle ABC and DF is drawn parallel to AC, meeting BC at E and also meeting the bisector of the exterior angle at C, at the point F. Prove that $DE = EF$.

93. The vertical angle of an isosceles triangle is 36°. Prove that the bisector of a base angle divides the triangle into two isosceles triangles.

94. In the previous exercise find the values of all the angles, and if a leg is a and the base b, determine the length of all the lines.

95. If one base of a trapezoid is double the other, the diagonals meet at a point which marks one-third the length of each diagonal.

96. If a polygon has n sides, prove that the number of its diagonals is $\frac{n(n-3)}{2}$.

97. If a polygon has n sides, how many re-entrant angles may it have?

98. The bisectors of exterior angles at A and B of a triangle ABC meet at D. Prove that $\angle D = 90^\circ - \frac{1}{2} C$.

99. In the triangle ABC, $\angle B > \angle C$ and BD is drawn to AC making $AD = AB$. Prove that $\angle ADB = \frac{1}{2}(B + C)$, and $\angle CBD = \frac{1}{2}(B - C)$.

100. BC is the hypotenuse of a right triangle ABC. AD is perpendicular to BC, and AE bisects the angle A. Prove that $\angle DAE = \frac{1}{2}(B - C)$.

101. BC is the hypotenuse of a right triangle ABC. AD is perpendicular to BC and AE bisects BC. Prove that $\angle DAE = (B - C)$.

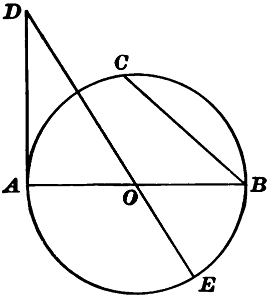
102. A median of a triangle is less than half the sum of the two adjacent sides.

BOOK II

THE CIRCLE

DEFINITIONS

130. A **Circle** is a plane figure bounded by a curved line, called the **Circumference,** every point of which is equally distant from a point within, called the centre; as *ABC*. The circumference is the locus of all points at a radius distance from the centre.



131. An **Arc** is any part of the circumference; as *AE*.

132. A **Chord** is a straight line joining the extremities of an arc; as *BC*.

133. A **Radius** is a line drawn from the centre to the circumference. The radii of a circle, or of equal circles, are equal; as *OA*, *OE*.

134. A **Diameter** is a chord passing through the centre. The diameters of a circle, or of equal circles, are equal, and a diameter is equal to two radii; as $AB = 2\,OB$. A diameter bisects a circle and also its circumference.

135. A **Secant** is a line which intersects the circumference of a circle in two points. A secant is often considered as drawn from a point outside the circle, sometimes from the intersection with some other line, to the concave circumference on the opposite side of the circle; as *DE*.

136. A **Tangent** is a line which touches but does not, though produced, intersect the circumference. A tangent is often considered as terminated at the point of contact; as *AD*.

137. A **Segment** is the part of a circle cut off by a chord; as the part included between the arc *BC* and the chord *BC*. A chord subtends two arcs and divides a circle into two segments. Unless otherwise stated, the lesser arc and the lesser segment are meant.

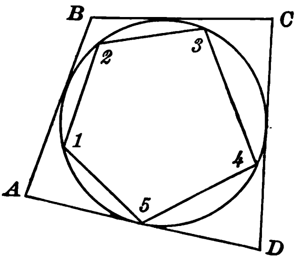
138. A **Sector** is the part of a circle included between two radii and the intercepted arc; as *EOB*.

139. A **Central Angle** is an angle between two radii; as *AOE*.

140. An **Inscribed Angle** is one whose sides are chords intersecting on the cicumference; as *ABC*. An angle is inscribed in a segment when its vertex is on the arc of the segment and its sides pass through the extremities of the chord of the segment.

141. An **Inscribed Polygon** is one whose sides are chords; as *12345*. The circumference is here *circumscribed* about the polygon.

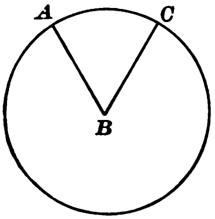
142. A **Circumscribed Polygon** is one whose sides are tangents; as *ABCD*. The circumference in this case is *inscribed* in the polygon.

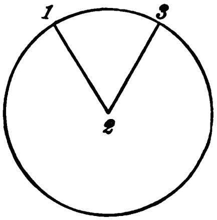


143. **Concentric Circles** have the same centre.

Proposition I

144. Theorem. *In the same circle, or in equal circles, equal central angles intercept equal arcs; and, conversely, equal arcs are intercepted by equal central angles.*





Appl.	$\left.\begin{matrix}AC\\13\end{matrix}\right\vert$ are = ⓢ. $B = 2$	
	Prove $AC = 13$	
Cons.	Place AB upon 12	
Dem.	BC falls upon 23	[$B = 2$]
	C “ “ 3	[radii =]
	AC “ “ 13	[$\overset{\frown}{s}$ of = ⓢ]
	$AC = 13$	[coincident throughout]

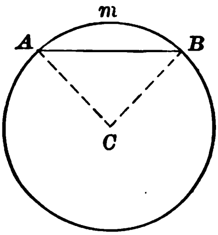
Conversely

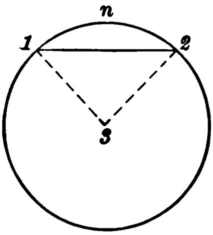
Appl.	$AC = 13$. Prove $B = 2$	
Cons.	Place AC upon 13	
Dem.	B falls upon 2	[centres of = $\overset{\frown}{s}$]
	AB “ “ 12	[coincident extremities]
	CB “ “ 32	
	$B = 2$	[common vertex, coincident sides]

145. Cor. In the same circle, or in equal circles, the greater of two central angles intercepts the greater arc, and conversely.

PROPOSITION II

146. Theorem. *In the same circle, or in equal circles, equal chords subtend equal arcs, and, conversely, equal arcs are subtended by equal chords.*





Appl. Let m represent the $\frown AB$

n " " $\frown 12$

$AB = 12$.

Prove $m = n$

Cons. Draw radii

Dem.

$$\left.\begin{array}{r}\triangle\text{s } ABC\\ 123\end{array}\right| = \left[\begin{array}{c}3\text{ s.} = 3\text{ s.}\\ \hline \text{radii} = (= \circledS)\\ \text{chords} = (\text{hyp.})\end{array}\right]$$

$C = 3$

$m = n$ [= cen. $\angle$s, ∴ = $\frown$s]

Conversely

Appl. $m = n$.

Prove $AB = 12$

Cons. Draw radii

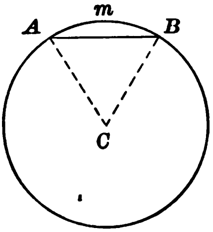
Dem. $C = 3$ [= $\frown$s, $(m = n)$ ∴ = $\angle$s]

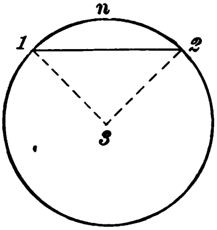
$$\left.\begin{array}{r}\triangle\text{s } ABC\\ 123\end{array}\right| = \left[\begin{array}{c}2\text{ s. and inc. } \angle\\ \hline \text{radii} = (= \circledS)\\ C = 3\end{array}\right]$$

$AB = 12$

PROPOSITION III

147. Theorem. *In the same circle, or in equal circles, the greater of two chords subtends the greater arc, and, conversely, the greater arc is subtended by the greater chord.*





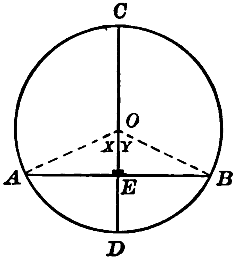
Appl.	$12 > AB$.	
	Prove $n > m$	
Cons.	Draw radii	
Dem.	$3 > C$	[In △s ABC and 123 2 s. = 2 s. (= radii) 3d s. un= (hyp.) ∴ inc. ∠ un=]
	$n > m$	[greater cent. ∠ intercepts greater arc]

Conversely.

Appl.	$n > m$.	
	Prove $12 > AB$	
Cons.	Draw radii	
Dem.	$3 > C$	[$n > m$ ∴ central ∠s un=]
	$12 > AB$	[In △s ABC and 123 2 s. = 2 s. (= radii) inc. ∠s un= ($3 > C$) ∴ 3d s. un=]

PROPOSITION IV

148. Theorem. *A radius perpendicular to a chord bisects the chord and the subtended arc.*



Appl. $OD =$ rad. $\perp AB$.
Prove $AE = BE$,
also $AD = BD$

Cons. Draw radii

Dem. $X = Y$ [In isos. $\triangle AOB$ $OE \perp$ base, $\therefore$ bis. base and vert. $\angle$]
$AE = BE$
$AD = BD$ [= cent. $\angle$s ($X = Y$), $\therefore$ = $\overset{\frown}{s}$]

149. Cor. I. DO produced = diameter.
$AC = BC$ [subtract AD and BD from = semi ⓢ]

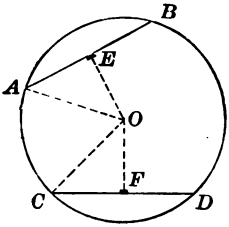
150. Cor. II. A line satisfying any two of the following conditions, satisfies all of them:

1. Passing through the centre.
2. Perpendicular to a chord.
3. Bisecting a chord.
4. Bisecting the lesser arc.
5. Bisecting the greater arc.

Prove the theorems obtained by selecting as an hypothesis any two of these conditions.

Proposition V

151. Theorem. *In the same or equal circles equal chords are equally distant from the centre.*



Appl. $AB = CD$.

Prove AB and CD = dist. from centre O

Cons. Draw radii OA, OC.

Draw ⊥s OE, OF

Dem. $AE = CF$ [halves of = s, (⊥s to chds.)]

△s AEO, CFO = [rt. △s, hyp. and s. $OA = OC$ (radii) $AE = CF$]

$OE = OF$

AB and CD = dist. [⊥s measure dist.]

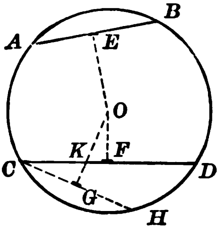
152. Cor. Conversely, chords which are equally distant from the centre are equal.

153. Sch. In subsequent theorems and exercises, two methods of proving chords equal are very frequently and almost equally used:

1. If they subtend equal arcs (Theorem II).
2. If they are equally distant from the centre of the circle (Corollary above).

PROPOSITION VI

154. Theorem. *Of two unequal chords, the less is more remote from the centre.*



Appl. $AB < CD$.
Prove AB more remote from centre than CD

Cons. Draw $CH = AB$.
Draw $\perp$s to three chords

Dem.
1. $OE = OG$ [= chds. = dist.]
2. $OK > OF$ [$\perp$ OF = least line.]
3. $OG > OK$ [OK = part of OG (H must be on arc CHD, G same side of CD as H)]

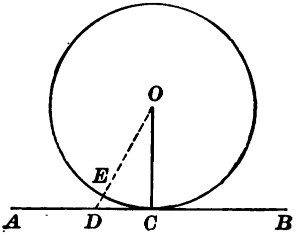
$OG > OF$ [combine (2) and (3)]
$OE > OF$ [subst. OE for OG (1)]
AB more remote than CD [$\perp$s un=]

155. Cor. I. Conversely, of chords unequally distant from the centre of a circle, the more remote is the less.

156. Cor. II. The diameter of a circle is greater than any other chord (it passes through the centre, *i.e.* is nearer the centre).

PROPOSITION VII

157. Theorem. *A perpendicular at the extremity of a radius is tangent to the circumference.*



Appl. AB is $\perp$ to rad. OC at C.
Prove $AB =$ tang. to $\odot$

Con. Draw OD, any line

Dem. $OC < OD$ [$\perp =$ least line]
D is outside $\odot$ [rad. OC and $OE =$]
$AB =$ tang. to $\odot$ [touches at C only]

158. Theorem. *Conversely, a tangent to a circumference is perpendicular to the radius drawn to the point of contact.*

Appl. $AB =$ tang. to $\odot$ at C.
Prove $OC = \perp$ to tang.

Con. Draw OD, to any pt. except C

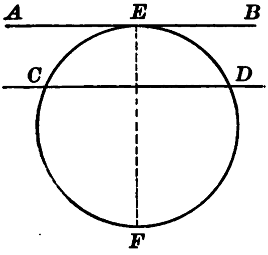
Dem. D is outside $\odot$ [$C =$ only pt. of contact]
$OC < OD$ [radii $=$]
$OC = \perp$ [$=$ least line]

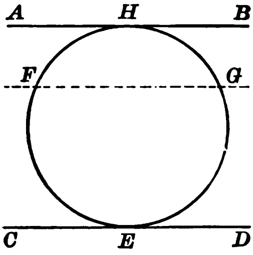
159. Cor. A perpendicular to a tangent, at the point of contact, passes through the centre of the circle.

PROPOSITION VIII

160. Theorem. *Parallel lines intercept equal arcs on circumference.*

1. *When one line is a secant and the other a tangent.*
2. *When both lines are tangents.*
3. *When both are secants.*





Case I

Appl. $AB =$ tang.
$CD =$ secant [∥ to each other]
Prove $CE = DE$

Cons. Draw diameter EF to point of contact

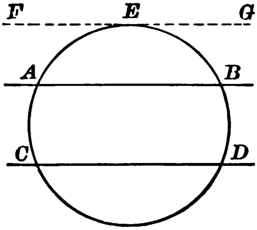
Dem. $EF \perp AB$ [rad. ⊥ tang.]
$EF \perp CD$ [⊥ both of two ∥s]
$CE = DE$ [rad. ⊥ chd. bisects $\frown$]

Case II

Appl. Both lines tangent.

Cons. Draw FG [a ∥ secant]

Dem. $FH = GH$ [*Case* I]
$FE = GE$ [*Case* I]
$HFE = HGE$ [addition]



Case III

Appl. Both lines secant

Cons. Draw FG [a ‖ tang.]

Dem. $CE = DE$ [*Case* I]

$AE = BE$ [*Case* I]

$AC = BD$ [subtraction]

EXERCISES

103. Tangents at the extremities of a diameter are parallel.

104. Perpendicular bisectors of the sides of an inscribed quadrilateral are concurrent.

105. Two intersecting chords which make equal angles with the diameter passing through their point of intersection are equal.

106. An inscribed trapezoid is isosceles, and an inscribed parallelogram is a rectangle.

107. If two circles are concentric, chords of the greater which are tangent to the less are equal.

108. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

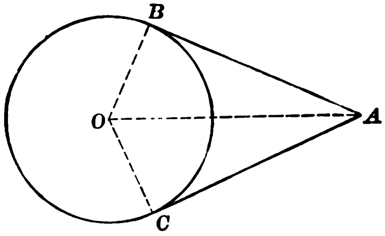
109. If equal chords intersect, their segments are respectively equal.

110. A straight line cannot intersect a circumference in more than two points.

111. If two circles intersect, the line joining their centres is less than the sum of their radii.

PROPOSITION IX

161. Theorem. *Two tangents from the same point to a circumference are equal.*



Appl. Prove tangents AB and AC equal

Cons. Draw radii to pts. of contact.
Connect AO

Dem. ABO and $ACO =$ rt. $\angle$s [tang. and rad. $\perp$]

rt. $\triangle\ ABO =$ rt. $\triangle\ ACO$
$AB = AC$

[hyp. and s.
AO com.
$BO = CO$ (radii)]

162. Cor. $\angle$s at A are $=$
$\angle$s at O are $=$

EXERCISES

112. In figure above AO bisects at rt. $\angle$s a line BC, joining points of contact.

113. Prove also that the angle BAC equals twice the angle OBC.

114. If two circles intersect, the line joining their centres bisects at right angles their common chord.

115. If two circles are tangent to each other, the line joining their centres passes through the point of contact.

116. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

117. A circumscribed parallelogram is a rhombus, and a circumscribed rectangle is a square.

MEASUREMENT

163. To **measure** a quantity is to find how many times it contains another quantity of the same kind, arbitrarily taken as a **unit** of measurement. Thus, the measurement of a straight line is the application of one of the **linear units,** a foot, a metre, etc.

164. The **numerical measure** of a quantity is the number which expresses how many times the quantity contains the unit; as 10 feet, 6 kilometres, etc.

165. The **ratio** between two quantities is their relative magnitude. The value of a ratio is the quotient of their numerical measures, as referred to the same unit of measure.

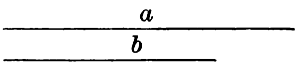
The ratio of a to b is written $\frac{a}{b}$, or $a : b$.

166. Commensurable quantities are those which have a *common measure.* Each of the quantities is, therefore, a multiple of this common measure. Thus, two lines, respectively $5\frac{1}{4}$ feet and $7\frac{2}{3}$ feet in length, are commensurable because each is a multiple of a line $\frac{1}{12}$ of a foot in length. $\frac{1}{12}$ is a common divisor (or measure) of $5\frac{1}{4}$ and $7\frac{2}{3}$, contained in the first 63 times and in the second 92 times. 63 and 92 are respectively the measures of the two lines, and the lines are commensurable.

167. Incommensurable quantities are those which have no common measure. No rational fraction can express exactly the value of the ratio of two incommensurables. However, by taking a fraction sufficiently small, a fraction may be found that differs from the exact value of the ratio by as little as we choose to specify.

Thus, let a and b be two lines such that

$$\frac{a}{b}=\frac{\sqrt{2}}{1} \text{ or } =\sqrt{2}$$



In attempting to express decimally the value of the ratio, we find that $\sqrt{2} = 1.4142135 +$, a value greater than 1.414213 and less than 1.414214.

Specifying, then, one millionth as the degree of accuracy, the value of the ratio $\frac{a}{b}$ is either

$$\frac{1414213}{1000000} \text{ or } \frac{1414214}{1000000}, \textit{ correct within } \frac{1}{1000000}.$$

The exact value of the ratio lies between these two fractions and can be expressed within any desired degree of accuracy. Any value of the ratio, except in the form $\sqrt{2}$, is only an approximate value, and the lines are incommensurable. The ratio between them is called an incommensurable ratio, and is a *constant* towards which the successive approximate values approach.

Expressed in general terms, if a and b are incommensurable, their ratio would be indicated by dividing b into n equal parts; a will contain one of these parts a certain number of times, such as m, with a remainder less than $\frac{1}{n}$. Therefore the value of the ratio is between $\frac{m}{n}$ and $\frac{m+1}{n}$, or $\frac{m}{n}$, correct within $\frac{1}{n}$.

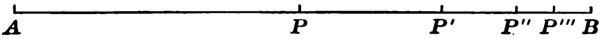
THE THEORY OF LIMITS

168. A **constant** is a quantity whose value is *fixed* throughout a given discussion.

169. A **variable** is a quantity whose value is not fixed, — a quantity which assumes different successive values, according to the conditions imposed upon it, during the discussion.

170. When the successive values of a variable, under the conditions imposed, approach a constant so that the difference between the variable and the constant becomes less than any specified quantity, however small, then the constant is called the **limit** of the variable. The variable is said to *approach indefinitely* its limit.

171. If a variable is increasing, it approaches a *superior* limit; if decreasing, an *inferior* limit.



Thus, a point may be supposed to move along a straight line from A toward B under this condition: that it shall first move one-half the distance from A to B; next, one-half the remaining distance; again, one-half the remaining distance, and so on *indefinitely*. P, P', P'', etc., indicate the successive positions of the moving point as it conforms to the condition of its motion in the discussion.

Evidently the point will approach B indefinitely, but will never reach it.

The distance AP, AP', etc., *i.e.* from A to the moving point, is a variable approaching AB as its *superior limit;* and the remaining distance PB, $P'B$, etc., decreases indefinitely and approaches *zero* as its *inferior limit.*

If the value of the first limit in this illustration is 2 inches, successive values of the variable will be 1 inch, $1\frac{1}{2}$ inches, $1\frac{3}{4}$ inches, etc.

Take the series

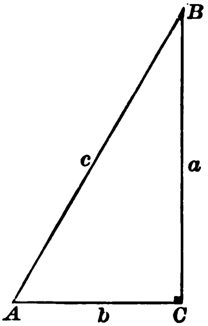
$$0.3,\ 0.33,\ 0.333,\ 0.3333,\ \cdots,$$

each successive value is a variable approaching the fraction $\frac{1}{3}$ as a limit.

As the number of sides of an equiangular polygon is increased the value of each interior angle approaches two right angles as a limit and the value of each exterior angle approaches zero as a limit.

In a right triangle ABC, let BC constantly increase, AC remaining fixed, then

$A =$ variable, approaching its limit, a rt. $\angle$,
$B =$ " " " " 0,
$\frac{a}{c} =$ " " " " 1,
$\frac{a}{b} =$ " " " " ∞,
$\frac{b}{c} =$ " " " " 0, etc.



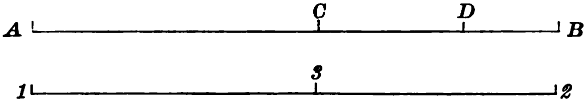
Or, let BC constantly decrease, then

$A =$ variable, approaching its limit 0,
$B =$ " " " " a rt. $\angle$,
$\frac{a}{c} =$ " " " " 0,
$\frac{b}{c} =$ " " " " 1, etc.

If an isosceles triangle has a fixed base and a constantly increasing altitude, the vertical angle approaches zero as a limit, each base angle approaches a right angle as a limit, and the two legs in their length approach infinity, and in their direction approach parallelism.

PROPOSITION X

172. Theorem. *If two variables are constantly equal and each approaches a limit, their limits are equal.*



Appl. AC and 13 are = variables approaching limits AB and 12. Prove $12 = AB$

INDIRECT PROOF. EXHAUSTION

Cons. If 12 does not $= AB$, suppose $12 < AB$
Apply 12 to AB, extending from A to D

Dem. 13 need not $= AC$ [If $12 < AB$, the variable AC may be $> AD$, while 13 must necessarily never exceed 12, or AD
Inequality of AC and 13 is contrary to hypothesis]

12 is not $> AB$ [similarly]
i.e. $12 = AB$

SECOND PROOF

Let the series 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$ approach 0.

If $\frac{x}{y}$ is a variable equal to 3, the successive values of the variable will be

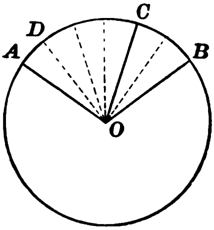
$$\frac{x}{y}, \quad \frac{x}{3y}, \quad \frac{x}{9y}, \quad \frac{x}{27y}, \quad \frac{x}{81y}, \quad \frac{x}{243y},$$

and will approach 0; *i.e.* equal variables approach equal limits.

173. Cor. If two variables have a constant ratio, their limits have the same ratio.

PROPOSITION XI

174. Theorem. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.*



Case I

Appl. When the arcs are commensurable.

Prove $\frac{AOB}{AOC}=\frac{AB}{AC}$

Cons. Apply AD, a common measure, to both arcs, and let it be contained in AB m times, and in AC n times.

Draw radii through pts. of = division

Dem. $\overset{\frown}{\text{s}}\ \frac{AB}{AC}=\frac{m}{n}$ [lines are as their numerical measures]

Small $\angle$s at $O =$ [$\overset{\frown}{\text{s}} =$]

i.e. $\angle AOD$ is a measure of

$\angle$s AOB and AOC

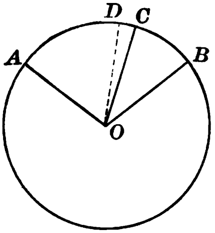
$\frac{\angle\text{s } AOB}{\angle\text{s } AOC}=\frac{m}{n}$ [$\angle$s are as their measures, m and n]

$\frac{AOB}{AOC}=\frac{AB}{AC}$ [both ratios $=\frac{m}{n}$]

Case II

Appl. When the arcs are incommensurable.

Prove $\frac{AOB}{AOC}=\frac{AB}{AC}$



Cons. Apply some measure of AB to AC, extending to D with remainder $DC <$ measure.

Draw radius OD

Dem. $\frac{AOB}{AOD}=\frac{AB}{AD}$ [*Case* I. (AB and AD are commens.)]

1. $AD=$variable, $AC=$its limit
2. $AOD=$ " $AOC=$ " "

[Indefinitely decrease measure, and ∴ indefinitely decrease DC ($DC <$ meas.)]

3. $\frac{AB}{AD}=$ " $\frac{AB}{AC}=$ " "
4. $\frac{AOB}{AOD}=$ " $\frac{AOB}{AOC}=$ " "

[Fractions are variables and limits respectively because of their denominators]

$\frac{AOB}{AOC}=\frac{AB}{AC}$ [variables =, ∴ their limits =]

175. Sch. Both the circumference of a circle and the angular space about the centre are divided into 360 equal parts called **degrees**, each degree into 60 equal parts called **minutes**, and each minute into 60 equal parts called **seconds**. Since central angles vary as their intercepted arcs, any central angle is the same fractional part of the angular space at the centre, as its intercepted arc is of the entire circumference. *A central angle has the same number of degrees, minutes, and seconds as its arc, and is said to be measured by its intercepted arc.*

PROPOSITION XII

176. Theorem. *An inscribed angle is measured by one-half its intercepted arc.*

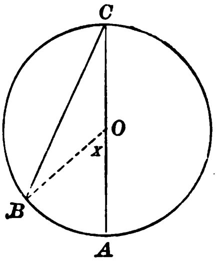


FIG. 1.

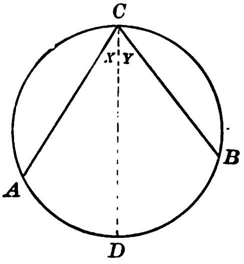


FIG. 2.

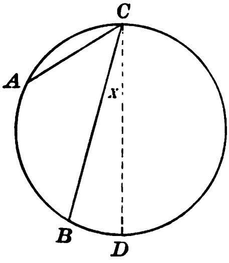


FIG. 3.

Appl. Prove that $\angle ACB$ is meas. by $\frac{1}{2}$ AB

Cons. *Case* I. When one side of the angle is a diameter.

Draw BO

Dem. $C = B = \frac{1}{2}(C + B)$ [BOC = isos. △ (= radii)]

$X = C + B$ [exterior ∠ = sum int. and opp. $\angle$s]

$C = \frac{1}{2}X$ [substitution]

C meas. by $\frac{1}{2}$ AB [X (central ∠) meas. by AB]

Case II. When the diameter is within the angle.

Hint. $ACB = X + Y$

Case III. When the diameter is without the angle.

Hint. $ACB = ACD - X$

177. Cor. I. *All angles inscribed in the same segment are equal* [meas. by ½ same arc]. Fig. 1.

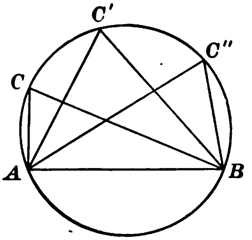


FIG. 1.

178. Cor. II. *An angle inscribed in a semicircle is a right angle* [meas. by ½ semicircumference]. Fig. 2.

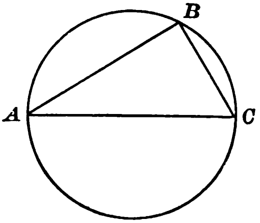


FIG. 2.

179. Cor. III. *The opposite angles of an inscribed quadrilateral are supplementary.*

$B+D=2$ rt. ∡s. [Together meas. by ½ the whole circumference, *i.e.* a semicircumference.] Fig. 3.

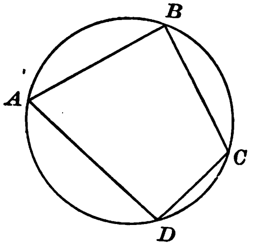


FIG. 3.

EXERCISES

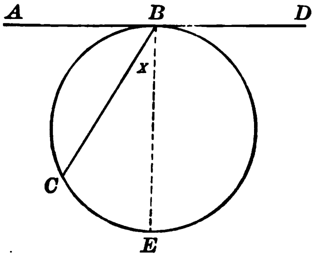
118. Tangents at the extremities of any two diameters form a rhombus.

119. A leg of a circumscribed trapezoid and lines drawn from its extremities to the centre of the circle form a right triangle.

120. If one leg of an isosceles triangle is the diameter of a circle, the circumference bisects the base.

PROPOSITION XIII

180. Theorem. *An angle between a tangent and a chord is measured by one-half its intercepted arc.*



Appl. Prove *ABC* is meas. by $\frac{1}{2}$ *BC*

Cons. Draw diam. *BE* to pt. of contact

Dem.	*ABE* meas. by $\frac{1}{2}$ *BCE*	[*ABE* = rt. ∠ (rad. ⊥ tang.) $\frac{1}{2}$ *BCE* = quadrant]
	x meas. by $\frac{1}{2}$ *CE*	[inscribed ∠]
	ABC meas. by $\frac{1}{2}$ *BC*	[*ABC* = (*ABE* − *x*)]
	DBC meas. by $\frac{1}{2}$ *BEC*	[similarly]

EXERCISES

121. In Fig. 1, on page 80, prove that bisectors of the angles C, C', and C'' are concurrent.

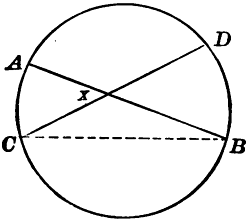
122. In Fig. 3, on page 80, if *AB*, *BC*, and *CD* are respectively arcs of 100°, 94°, and 55°, how many degrees in each of the four angles of the quadrilateral?

123. In the figure of Theorem XIII, if *X* is an angle of 32°, how many degrees in the arc *BC*?

124. In the same figure, if the arc *CE* is $\frac{1}{6}$ of the circumference, what is the ratio between the angles *ABC* and *DBC*?

Proposition XIV

181. Theorem. *An angle between two chords intersecting within the circumference is measured by one-half the sum of the intercepted arcs.*



Appl. Prove $x = \frac{1}{2}(AC + DB)$

(*i.e.* = *in degrees*, or is measured by)

Cons. Draw CB

Dem. $x = B + C$ [ext. ∠ = sum int. and opp. ⦤s]

$x = \frac{1}{2}(AC + DB)$ $\left[\begin{matrix} B = \frac{1}{2} AC \text{ (inscribed } \angle) \\ C = \frac{1}{2} DB \text{ (inscribed } \angle) \end{matrix}\right]$

EXERCISES

125. In the figure above if X = an ∠ of 42° and B = an ∠ of 18°, how many degrees in the arc DB?

126. In the same figure if AD = an arc of 123° and X = an ∠ of 36°, how many degrees in the arc CB?

127. Two angles of an inscribed triangle are measured by $\frac{1}{4}$ and $\frac{1}{8}$ of the circumference, respectively. How many degrees in each angle of the triangle?

128. If the three angles of an inscribed triangle are in the ratio of 1 : 3 : 5, how many degrees in each of the arcs subtended by the sides of the triangle?

129. If either base angle of an inscribed isosceles triangle is 40°, what is the ratio of the arc subtended by the base of the triangle to the arc subtended by either leg?

Proposition XV

182. Theorem. *An angle between two tangents, a tangent and a secant, or two secants, intersecting without the circumference is measured by one-half the difference of the intercepted arcs.*

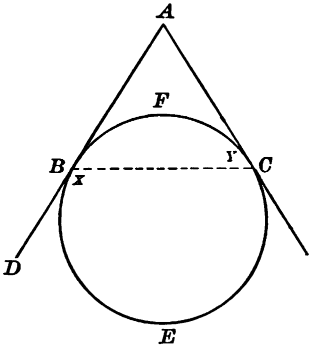


Fig. 1.

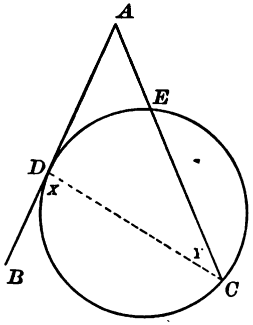


Fig. 2.

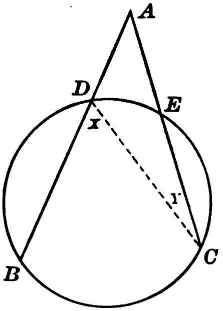


Fig. 3.

Appl. *Case* I. Prove $A = \frac{1}{2}(BEC - BFC)$
Case II. " $A = \frac{1}{2}(DC - DE)$
Case III. " $A = \frac{1}{2}(BC - DE)$

Cons. Draw BC (Fig. 1) and DC (Fig. 2) and (Fig. 3)

Dem. Hint: (for all cases)

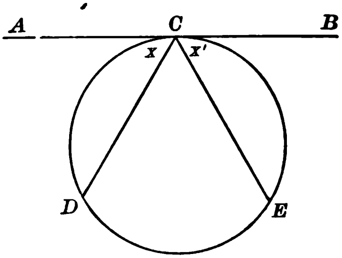
$$X = A + Y$$
$$A = X - Y$$

EXERCISES

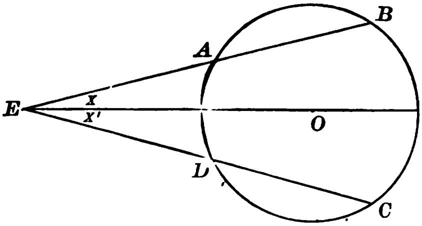
130. If, in Fig. 1 above, the arc BFC is two-fifths of the circumference, how many degrees are there in the angles A, X, and Y?

131. If, in Fig. 3 above, the angle $Y = 25°$ and the arc $BC = 105°$, how many degrees are there in $\angle A$, and in the sum of the arcs BD and EC?

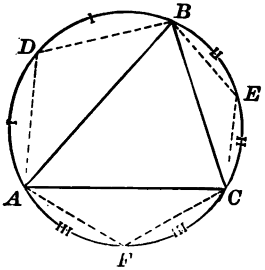
132. Prove that the angle between two tangents is the supplement of the angle between two radii drawn to the points of contact.



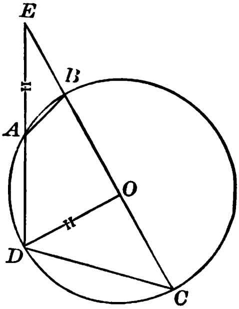
Ex. 141. $AB =$ tang.
Prove $CD = CE$



Ex. 142. $EO =$ secant through centre.
Prove $AB = DC$. What kind of a figure is $ABCD$?



Ex. 143. Prove $D + E + F =$ 4 rt. $\angle$s



Ex. 144. Prove $\angle DOC = 3\,E$

EXERCISES

145. If each base angle of an inscribed isosceles triangle is double the vertical angle, tangents to the circle at the three vertices of the triangle will form an isosceles triangle in which each base angle is one-third the vertical angle.

146. AB is the diameter of a circle and CD a chord perpendicular to AB; E is any point in the arc BC, and AE cuts CD at F. Prove that $\angle EFD = \angle ACE$.

147. Two circles are tangent internally, and the radius of one equals the diameter of the other. AB is a common diameter, and CD is a chord of the larger circle drawn through the centre of the smaller, perpendicular to AB and cutting the smaller circumference at E and F. GH is a second chord of the larger circle, perpendicular to CD at E or F. Prove that the greater segments of the chords CD and GH are equal, also the lesser segments.

148. The bisector of the angle between two tangents passes through the centre of the circle.

149. The bisectors of the angles of a circumscribed quadrilateral are concurrent.

150. The bisectors of the angles of an inscribed triangle meet at D. If AD produced meets the circumference at E, prove that $ED = EC$.

151. In a circle whose centre is A are drawn two circles with centres B and C which touch each other and also the outer circumference. Prove that the perimeter ABC equals the diameter of the outer circle.

152. ABC is a triangle inscribed in a circle whose centre is O, and OD is drawn perpendicular to AC. Prove that $\angle AOD = \angle B$.

153. If a circle is inscribed in a right triangle, the sum of the legs equals the hypotenuse plus two radii.

154. AO and BO are radii of a circle perpendicular to each other. AC is a chord intersecting OB at D. If a tangent at C meets OB produced at E, prove that DEC is an isosceles triangle.

155. A circle inscribed in the triangle ABC touches the sides AB, BC, and CA at D, E, and F, respectively. Prove that $\angle DEF = 90^\circ - \frac{1}{2} A$.

156. AB is a common chord of two intersecting circles, and AC and AD are the diameters of the two circles. Prove that CD passes through B.

157. AB and AC are tangents to a circle, and D any point in the greater arc BC. Prove that the sum of the angles ABD and ACD is constant.

158. AB and AC are tangents to a circle. Through D, any point in the lesser arc BC, is drawn a tangent meeting the first two tangents at E and F. Prove that the perimeter of the triangle AEF is constant, because equal to the sum of AB and AC, and that the angle BOC is twice the angle EOF, O being the centre of the circle.

159. If two opposite sides of an inscribed quadrilateral are produced to meet, and a perpendicular be drawn to the bisector of the angle between them from the point of intersection of the diagonals, prove that the perpendicular bisects the angle between the diagonals.

160. A base angle of an inscribed isosceles triangle equals one-third a right angle. Prove that a leg of the triangle equals the radius of the circle.

Loci

161. What is the locus of points at a given distance from a given circumference? a radius distance from a given circumference?

162. What is the locus of the middle points of a series of parallel chords in a circle?

163. What is the locus of the middle points of equal chords in a circle?

164. What is the locus of the middle points of all chords drawn from a given point on the circumference?

165. A straight line moves so that its two ends constantly touch two indefinite lines which are perpendicular to each other. What is the locus of the middle point of the moving line?

166. What is the locus of the vertex of the right angle of a right triangle which has a given hypotenuse?

167. What is the locus of the centres of circles tangent to two intersecting straight lines?

168. What is the locus of the centres of circles tangent to a given circle at a given point?

BOOK III

PROPORTION — SIMILAR POLYGONS — REGULAR POLYGONS

DEFINITIONS

183. A **Proportion** is an equality of ratios. Four quantities are said to be in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth.

A proportion may be written in either of the following forms:

$$a : b = c : d, \text{ or } a : b :: c : d, \text{ or } \frac{a}{b} = \frac{c}{d}$$

184. The first and fourth terms of a proportion are called the **extremes**, and the second and third terms the **means.**

The first and third terms are called the **antecedents,** and the second and fourth terms the **consequents.**

Either ratio is called a **couplet.**

In a proportion in which the means are equal, either mean is called a **mean proportional** between the other two, and the last term is called a **third proportional** to the first and second. If the means are unequal, the last term of a proportion is, naturally, a **fourth proportional.**

185. A **Continued Proportion** is a series of equal ratios; as,

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$$

PROPOSITION I

186. Theorem. *In a proportion the product of the extremes is equal to the product of the means.*

$$a : b = c : d$$

or $$\frac{a}{b} = \frac{c}{d}$$

$$ad = bc \qquad \text{[clearing of fractions]}$$

187. Sch. A proportion is an equation, and many principles used in the transformation of equations, such as squaring the equation, are applicable to proportions.

PROPOSITION II

188. Theorem. *If the product of two quantities is equal to the product of two others, the factors of either product may be made the extremes, and the factors of the other product the means of a proportion.*

$$ad = bc$$

$$\frac{a}{b} = \frac{c}{d} \qquad \left[\text{divide by } bd\right]$$

or $$\frac{a}{c} = \frac{b}{d} \qquad \left[\text{divide by what ?}\right]$$

or $$\frac{d}{b} = \frac{c}{a}, \text{ etc.} \qquad \left[\text{divide by what ?}\right]$$

PROPOSITION III

189. Theorem. *In a proportion the terms are in proportion by Alternation; i.e. the first term is to the third as the second is to the fourth.*

$$a : b = c : d$$

$$ad = bc \qquad [?]$$

$$\frac{a}{c} = \frac{b}{d} \qquad \left[\text{divide by what ?}\right]$$

or $$a : c = b : d$$

PROPOSITION IV

190. Theorem. *In a proportion the terms are in proportion by Inversion; i.e. the second term is to the first as the fourth is to the third.*

$$a : b = c : d$$

$$ad = bc \qquad [?]$$

$$\frac{d}{c} = \frac{b}{a} \qquad \left[\text{divide by what ?}\right]$$

or

$$b : a = d : c$$

PROPOSITION V

191. Theorem. *In a proportion the terms are in proportion by Composition; i.e. in the two couplets, the sums of antecedent and consequent are compared with either the antecedents or with the consequents.*

$$a : b = c : d$$

(1) or

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b} + 1 = \frac{c}{d} + 1 \qquad \left[\text{add 1 to each member}\right]$$

$$\frac{a+b}{b} = \frac{c+d}{d} \qquad [?]$$

or

$$\frac{b}{a} = \frac{d}{c} \qquad \left[\text{(1) by inversion}\right]$$

$$\frac{b}{a} + 1 = \frac{d}{c} + 1$$

$$\frac{a+b}{a} = \frac{c+d}{c}$$

Proposition VI

192. Theorem. *In a proportion the terms are in proportion by Division; i.e. in the two couplets the differences of antecedent and consequent are compared with either the antecedents or with the consequents.*

$$a : b = c : d$$

or

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b} - 1 = \frac{c}{d} - 1 \qquad \text{[subtract 1 from each member]}$$

$$\frac{a-b}{b} = \frac{c-d}{d}$$

$$\frac{a-b}{a} = \frac{c-d}{c} \qquad \text{[similarly]}$$

193. Cor. In any proportion the terms are in proportion by composition and division.

$$a + b : a - b = c + d : c - d \qquad \text{[transform and combine results of last two theorems]}$$

Proposition VII

194. Theorem. *Both terms of either couplet, or both antecedents, or both consequents, of a proportion may be multiplied or divided by any quantity and the resulting quantities will be in proportion.*

$$a : b = c : d$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{ma}{mb} = \frac{c}{d}$$

$$\frac{ma}{b} = \frac{mc}{d}$$

$$\frac{a}{bx} = \frac{c}{dx}$$

[multiplying as in any equations of *fractions*]

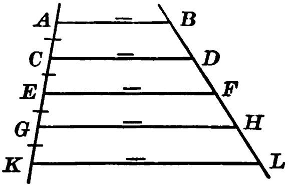
PROPOSITION VIII

195. Theorem. *In a continued proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let	$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=r$	[r = value of each ratio]
	$a=br$ $c=dr$ $e=fr$	[clearing of fractions separately]
	$a+c+e=(b+d+f)r$	[add]
	$\frac{a+c+e}{b+d+f}=r$	[divide by $(b+d+f)$]
	$\frac{a+c+e}{b+d+f}=\frac{a}{b}$	[substitute for r, $\frac{a}{b}$ or $\frac{c}{d}$, etc.]

PROPOSITION IX

196. Theorem. *If a series of parallels intercept equal distances on one of two transverse lines they intercept also equal distances on the other.*



Appl. $| = | = |$, etc.

Prove $BD = DF = FH$, etc.

Dem.	$BD = DF$	[in trapezoid $ABFE$, CD bisects one leg and is $\parallel$ to the bases]
	$DF = FH$	[similarly in trapezoid $CDHG$]
	$BD = DF = FH$, etc.	[Ax. 1]

PROPOSITION X

197. Theorem. *A line parallel to one side of a triangle divides the other two sides proportionally.*

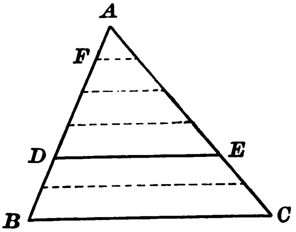


FIG. 1.

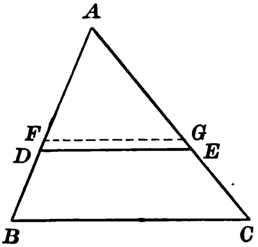


FIG. 2.

Case I (Fig. 1)

Appl. When the side AB and its segment are commensurable.

$DE \parallel BC$

Prove $\frac{AB}{AD}=\frac{AC}{AE}$

Cons. Apply AF, a common measure, to AB and AD, and let it be contained in AB m times, and in AC n times.

Through points of equal division draw parallels to BC

Dem. $\frac{AB}{AD}=\frac{m}{n}$ [two lines are as their numerical measures]

$\frac{AC}{AE}=\frac{m}{n}$ [$\parallel$s intercepting = distances on one transverse line and therefore on the other, m on AC, n on AE]

$\frac{AB}{AD}=\frac{AC}{AE}$ [= to same ratio $\frac{m}{n}$]

Case II (Fig. 2)

Appl. When AB and AD are incommensurable

Cons. Apply some measure of AB to AD, contained in AD a certain number of times, to the point F, with remainder FD, $<$ measure.

Draw $FG \parallel BC$

Dem. $\frac{AB}{AF}=\frac{AC}{AG}$ $\left[\begin{array}{l}\text{by } \textit{Case} \text{ I, } AB \text{ and } AF \text{ commens.}\\ \text{by construction}\end{array}\right]$

1. $AF=$ variable and $AD=$ its limit $\left[\begin{array}{l}\text{indefinitely decrease}\\ \text{measure, and}\\ \therefore \text{ indefinitely decrease}\\ FD\end{array}\right]$

2. $AG=$ " " $AE=$ " " [similarly]

3. $\frac{AB}{AF}=$ " " $\frac{AB}{AD}=$ " " [?]

4. $\frac{AC}{AG}=$ " " $\frac{AC}{AE}=$ " " [?]

$\frac{AB}{AD}=\frac{AC}{AE}$ $\left[\begin{array}{l}\text{variables (3) and (4)} = (\textit{Case} \text{ I})\\ \therefore \text{ limits (3) and (4)} = \text{(Theory of Limits)}\end{array}\right]$

198. **Cor.** (1) $AB:AD=AC:AE$ [theorem]

$AB-AD:AD=AC-AE:AE$ [division]

or $DB:AD=EC:AE$

(2) $AB:DB=AC:EC$ [similarly]

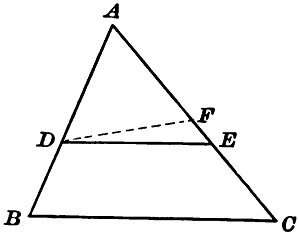
$\frac{AB}{AC}=\frac{AD}{AE}=\frac{DB}{EC}$ $\left[\begin{array}{l}\text{(1) and (2) by}\\ \text{alternation, etc.}\end{array}\right]$

and in general, if a line, or if any number of lines, be drawn parallel to a side of a triangle, the corresponding segments of the other two sides are proportional.

Note. — Hereafter, in the demonstration of theorems, the minor reasons for the principal or important steps of proof will often be omitted, and the student is expected to supply the reasons, as required.

PROPOSITION XI

199. Theorem. *Conversely, a line which divides two sides of a triangle proportionally is parallel to the third side.*



Appl. $\frac{AB}{AD}=\frac{AC}{AE}$

Prove $DE \parallel BC$

INDIRECT PROOF — COINCIDENCE

Cons. If DE is not $\parallel$

through D draw $DF \parallel BC$

Dem. $\frac{AB}{AD}=\frac{AC}{AE}$ [hypothesis]

$\frac{AB}{AD}=\frac{AC}{AF}$ [$DF \parallel BC$ (cons.)]

$AE=AF$ [fourth terms of = prop'ns]

E coincides with F

DE coincides with DF

$DE \parallel BC$ [$DF \parallel BC$ (cons.)]

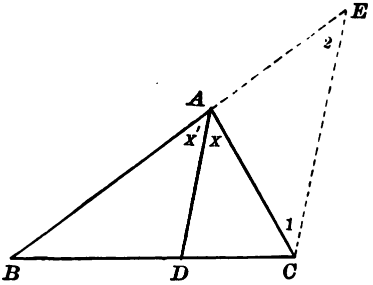
---

169. If two parallels to AC, a side of a triangle ABC, meet the side AB at D and F, and the side BC at E and G, respectively, prove

$$\frac{DF}{FA}=\frac{EG}{GC}=\frac{DA}{EC}$$

PROPOSITION XII

200. Theorem. *The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*



Appl. $x = x'$

Prove $\dfrac{DB}{DC} = \dfrac{AB}{AC}$

Cons. Draw $CE \parallel$ to DA
meeting BA produced at E

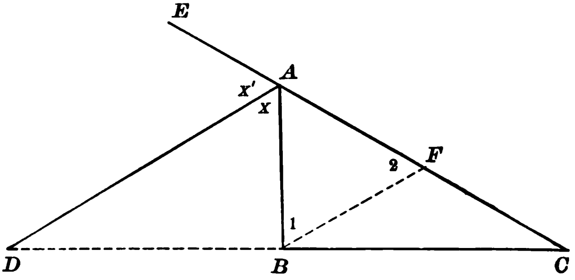
Dem.

$1 = x$	[alt. int. ∠s]
$2 = x'$	[corres. ∠s]
$1 = 2$	[= to = ∠s]
$AEC =$ isos. △	[base ∠s =]
$\dfrac{DB}{DC} = \dfrac{AB}{AE}$	[$DA \parallel CE$ in △ BEC]
$\dfrac{DB}{DC} = \dfrac{AB}{AC}$	[subs. AC for its = AE (legs of isos. △)]

201. Sch. By the *segments* of a line are meant the distances from a point of division on the line to the extremities of the line. The point of division may be on the line or on the line produced, forming respectively internal and external segments. In the former case the line equals the sum of its segments, and in the latter the difference.

PROPOSITION XIII

202. Theorem. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*



Appl. $X = X'$ Prove $\frac{DB}{DC} = \frac{AB}{AC}$

Cons. Draw $BF \parallel$ to DA

Dem. $1 = X$ [alt. int. $\angle$s]

$2 = X'$ [?]

$1 = 2$

$ABF =$ isos. $\triangle$

$\frac{DB}{DC} = \frac{AF}{AC}$ $\left[BF \parallel DA\right]$

$\frac{DB}{DC} = \frac{AB}{AC}$ [subs. AB for its $= AF$]

203. Sch. If a line is divided so that the ratio between its internal segments equals the ratio between its external segments, the line is said to be divided *harmonically*.

The bisector of an angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side of the triangle harmonically, because the ratios between the segments of this side, both internal and external, equal the ratio between the adjacent sides.

SIMILAR POLYGONS

204. Definition. Two polygons are **similar** when they are mutually equiangular and have their homologous sides proportional.

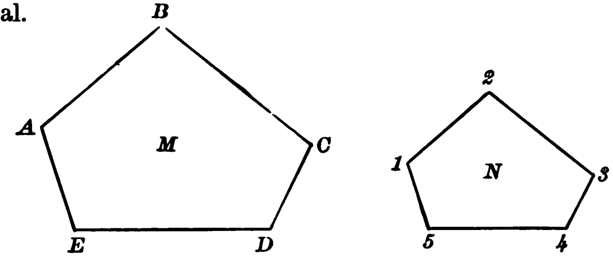


FIG. 1.

M and *N* (Fig. 1) are similar if $A = 1$, $B = 2$, $C = 3$, etc., and $\frac{AB}{12} = \frac{BC}{23} = \frac{CD}{34}$, etc.

M and *N* (Fig. 2) are *not* similar, even if $A = 1$, $B = 2$, etc. unless $\frac{AE}{15} = \frac{ED}{54}$, which is evidently not true.

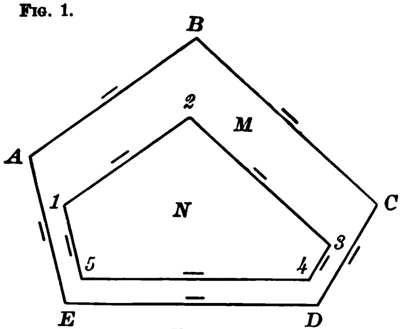


FIG. 2.

M and *N* (Fig. 3) are *not* similar, even if the homologous sides are proportional, because the homologous angles are not equal.

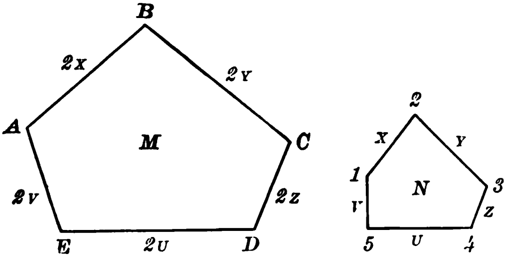
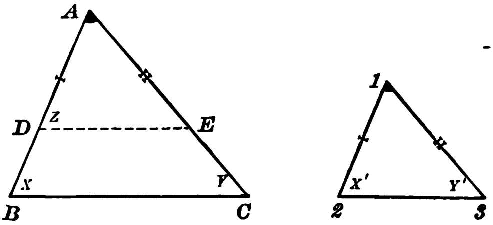


FIG. 3.

205. Similar figures have the same form or shape. The ratio between homologous sides is the *ratio of similitude* of the figures.

PROPOSITION XIV

206. Theorem. *Two triangles are similar if they are mutually equiangular.*



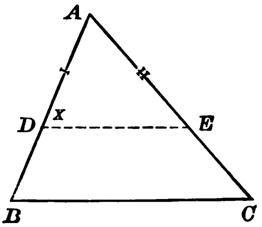
Appl.	$\left.\begin{matrix}ABC\\ 123\end{matrix}\right\vert$ mut. $=\angle$	
	Prove homologous sides proportional	
Cons.	Measure $AD = 12$	
	$AE = 13$	
	Join DE	
Dem.	$\left.\begin{matrix}\triangle ADE\\ 123\end{matrix}\right\vert =$	[2 s. and inc. $\angle$]
	$Z = X'$	
	$Z = X$	[$X = X'$ (hyp.)]
	$DE \parallel BC$	[corresp. $\angle$s $=$]
	$\frac{AB}{AD} = \frac{AC}{AE}$	[$DE \parallel$ to BC]
or	$\frac{AB}{12} = \frac{AC}{13}$	[subs. 12 for AD 13 for AE]
	$\frac{AB}{12} = \frac{BC}{23}$	[similarly]

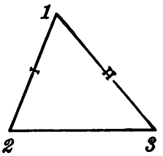
---

170. In a triangle ABC, $AB = 6$, $BC = 10$, and $CA = 12$. Find the segments of CA, formed by a bisector of the angle B.

PROPOSITION XV

207. Theorem. *Two triangles are similar if the homologous sides are proportional.*





Appl. $\frac{AB}{12} = \frac{AC}{13} = \frac{BC}{23}$

Prove $\triangle$ similar

Cons. Measure $AD = 12$

$AE = 13$

Join DE

Dem. $\frac{AB}{AD} = \frac{AC}{AE}$ [subs. in given proportion acc. to construction]

$DE \parallel BC$ [sides divided proportionally]

$X = B$

$\triangle \begin{array}{c|} ABC \\ ADE \end{array}$ sim. [mut. $= \angle$]

$\frac{AB}{AD} = \frac{BC}{DE}$ [homologous sides proportional]

$\frac{AB}{12} = \frac{BC}{23}$ [hypothesis]

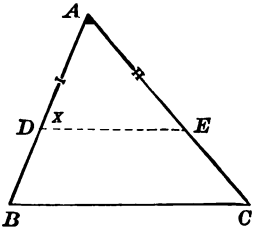
$DE = 23$ [fourth terms of = proportions]

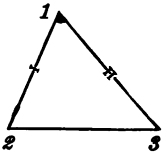
$\triangle \begin{array}{c|} ADE \\ 123 \end{array}$ = [3 s. = 3 s.]

$\triangle \begin{array}{c|} ABC \\ 123 \end{array}$ sim. [subs. 123 for its $= ADE$]

PROPOSITION XVI

208. Theorem. *Two triangles are similar if an angle of one equals an angle of the other and the including sides are proportional.*





Appl. $A = 1$

$$\frac{AB}{12} = \frac{AC}{13}$$

Prove △ similar

Cons. Measure $AD = 12$

$AE = 13$

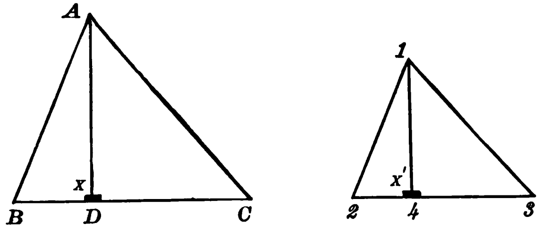
Join DE

Dem.

$\triangle\ \begin{array}{c\|}ADE\\123\end{array} =$	[2 s. and inc. ∠]
$\frac{AB}{AD} = \frac{AC}{AE}$	[subst. | for | || for ||]
$DE \parallel BC$	[divides s. prop'ly]
$x = B$	
$\begin{array}{c\|}ABC\\ADE\end{array}$ sim.	[mut. = ∠]
$\begin{array}{c\|}ABC\\123\end{array}$ sim.	[subs. 123 for ADE]

PROPOSITION XVII

209. Theorem. *The homologous altitudes of similar triangles are in the same ratio as any two homologous sides.*



Appl. AD and $1\,4$ are homol. altitudes.

Prove $\frac{AD}{1\,4}=\frac{AB}{1\,2}=\frac{BC}{2\,3}$, etc.

Dem. $\triangle s\ \frac{ABD}{1\,2\,4}$ | sim. $\left[\frac{\text{mut.} = \angle}{\begin{matrix} B = 2 \text{ (hyp.)} \\ X = X' \text{ (rt. } \angle s) \end{matrix}}\right]$

$\frac{AD}{1\,4}=\frac{AB}{1\,2}$

$\frac{AB}{1\,2}=\frac{BC}{2\,3}$, etc. [in given sim. △s]

$\frac{AD}{1\,4}=\frac{BC}{2\,3}$, etc.

EXERCISES

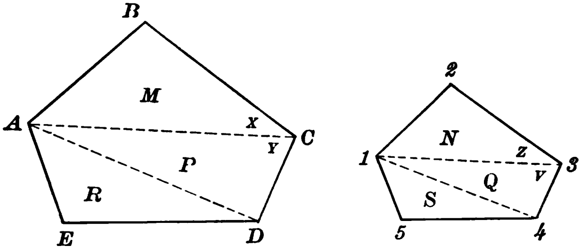
171. In a triangle ABC, $AB = 40$, $BC = 50$, and $CA = 70$. Find the external segments of CA, formed by a bisector of the exterior angle at B.

172. In the diagram of Prop. XI, if $DB : DA = 4 : 5$ and $AC = 36$, what is the length of EC?

173. In Prop. XIII, if AB, AC, and DB are 7, 9, and 12, respectively, what is BC?

PROPOSITION XVIII

210. Theorem. *Two polygons are similar if they are composed of the same number of triangles similar each to each and similarly placed.*



Appl. M sim. N P sim. Q R sim. S

Prove ⓢ sim.

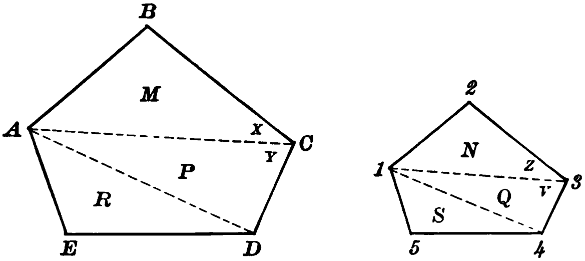
Dem.	$B = 2$	[in sim. △s M and N]
	$X = Z$	[" " " " "]
	$Y = V$	[" " P " Q]
	$C = 3$	[$(X+Y)=(Z+V)$]
	$\frac{AB}{12} = \frac{BC}{23}$	[in sim. △s M and N]
	$\frac{AC}{13} = \frac{BC}{23}$	[" " " " "]
	$\frac{AC}{13} = \frac{CD}{34}$	[" " P " Q]
	$\frac{BC}{23} = \frac{CD}{34}$	[Ax. I]
	ⓢ sim.	[mut. = ∠ and hom. sides prop.]

211. Cor. I. Homologous diagonals in similar polygons have the same ratio as any two homologous sides.

212. Cor. II. Homologous lines, of any kind, in similar polygons have the ratio of similitude of the polygons.

Proposition XIX

213. Theorem. *Two similar polygons may be decomposed into the same number of triangles, similar each to each and similarly placed.*

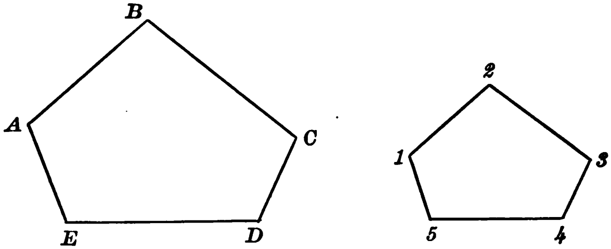


Appl. Prove *M* sim. *N*, etc., in given sim ⬠

Dem.	$B = 2$	[in sim. ⬠]
	$\frac{AB}{12} = \frac{BC}{23}$	[" "]
	M and *N* sim.	[$\angle = \angle$ and including s. prop.]
	$C = 3$	[in sim. ⬠]
	$X = Z$	[" △ *M* and *N*]
	$Y = V$	[subtract = from =]
	$\frac{AC}{13} = \frac{BC}{23}$	[in sim. △ *M* and *N*]
	$\frac{CD}{34} = \frac{BC}{23}$	[" " ⬠]
	$\frac{AC}{13} = \frac{CD}{34}$	[Ax. I]
	P and *Q* sim. etc.	[$\angle = \angle$ and including s. prop.]

PROPOSITION XX

214. Theorem. *The perimeters of similar polygons are to each other as any two homologous sides.*



Appl. Prove $\dfrac{AB + BC + \text{etc.}}{12 + 23 + \text{etc.}}$ or $\dfrac{A - E}{1 - 5} = \dfrac{AB}{12}$, etc.

Dem. $\dfrac{AB}{12} = \dfrac{BC}{23} = \dfrac{CD}{34} =$ etc. [homol. s. of sim. Ⓢ]

$$\frac{AB + BC + CD, \text{ etc.}}{12 + 23 + 34 + \text{etc.}} = \frac{AB}{12}, \text{ etc.}$$

[in a series of = ratios, sum of antecedents is to sum of consequents as any antecedent is to its consequent]

i.e. $\dfrac{A - E}{1 - 5} = \dfrac{AB}{12} = \dfrac{BC}{23}$, etc.

EXERCISES

174. The perimeters of two similar polygons are 76 and 114. If a diagonal of the first is 16, what is the homologous diagonal of the second?

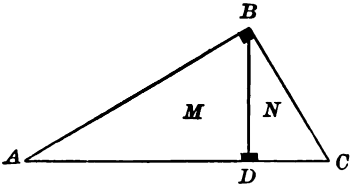
175. A tree casts a shadow 100 feet long, and at the same time a vertical rod 6 feet high casts a shadow 8 ft. 4 in. long. What is the height of the tree?

176. In Prop. XIII, if $DA : BF = 3 : 1$, what is the ratio of DB to BC?

PROPOSITION XXI

215. Theorem. *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse,*

1. *The triangles formed are similar to the whole triangle and to each other.*
2. *The perpendicular is a mean proportional between the two segments of the hypotenuse.*
3. *Either side about the right angle is a mean proportional between the whole hypotenuse and the adjacent segment.*



Dem. 1. M sim. ABC $\left[\frac{\text{mut.} = \angle}{A \text{ com. and each has a rt. } \angle}\right]$

N " " [similarly]

M " N [both sim. ABC]

2. $AD : BD = BD : DC$ [sim. $\triangle$s M and N]

3. $AC : AB = AB : AD$ [sim. $\triangle$s M and ABC]

or $AC : BC = BC : DC$ [" " N " "]

216. Cor. I. 1. $\overline{BD}^2 = AD \times DC$ [prod. ext. = prod. means (*Case* II)]

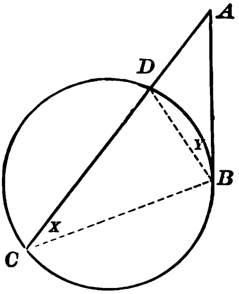
2. $\overline{AB}^2 = AC \times AD$ [prod. ext. = prod. means (*Case* III)]

3. $\overline{BC}^2 = AC \times DC$ [prod. ext. = prod. means (*Case* III)]

Cor. II. $\frac{\overline{AB}^2}{\overline{BC}^2} = \frac{AD}{DC}$ [combine (2) and (3)]

PROPOSITION XXII

217. Theorem. *If a tangent and a secant are drawn from a point to a circumference, the tangent is a mean proportional between the secant and its external segment.*



Appl. Prove $AC : AB = AB : AD$

Cons. Draw BD and BC

Dem. $X = Y$ [meas. by $\frac{1}{2}$ BD]

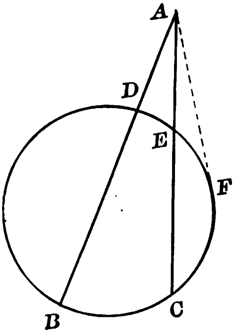
$\triangle$ ABC, ABD sim. $\left[\dfrac{\text{mut.} = \angle}{\begin{array}{c} A \text{ com.} \\ X = Y \end{array}}\right]$

$AC : AB = AB : AD$

218. Cor. I. $\overline{AB}^2 = AC \times AD.$

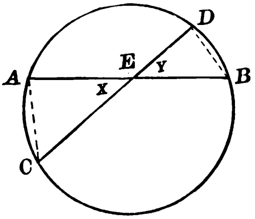
219. Cor. II. If two or more secants are drawn from a point to a circumference, the product of any secant and its external segment equals the product of any other and its external segment.

Dem. $AB \times AD = AC \times AE$ $[= \overline{AF}^2]$



Proposition XXIII

220. Theorem. *If two chords intersect within a circle, the product of the segments of one equals the product of the segments of the other.*



Dem. $X = Y$

$A = D$

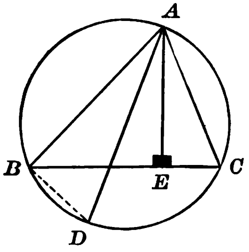
$$\triangle\ \left.\begin{matrix}AEC\\DEB\end{matrix}\right|\ \text{sim.}$$

$AE : CE = DE : BE$

$AE \times BE = CE \times DE$

Proposition XXIV

221. Theorem. *The product of two sides of a triangle equals the product of the diameter of the circumscribed circle and the perpendicular to the third side from the opposite vertex.*



Dem. $ABD = \text{rt.} \angle$ [insc. in semi ⊙]

$C = D$ [" " same segment]

$$\triangle\ \left.\begin{matrix}ABD\\AEC\end{matrix}\right|\ \text{sim.}$$

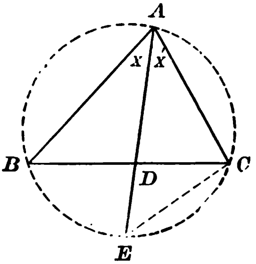
$AB : AD = AE : AC$

$AB \times AC = AD \times AE$

222. Cor. The diameter of a circle circumscribed about a triangle equals what?

PROPOSITION XXV

223. Theorem. *The product of two sides of a triangle equals the product of the segments of the third side formed by the bisector of the opposite angle plus the square of the bisector.*



Appl. Prove $AB \times AC = BD \times CD + AD^2$

Cons. Circumscribe ⊙ about △.
Produce AD to E. Draw EC

Dem. $\triangle\!\!\triangle\ \begin{array}{r|l} ABD & \\ AEC & \end{array}$ sim. $\left[\dfrac{\text{mut.} = \angle}{\begin{array}{c} x = x' \text{ (hyp.)} \\ B = E \text{ (insc. in same seg.)} \end{array}}\right]$

$$\begin{aligned} AB : AD &= AE : AC \\ AB \times AC &= AD \times AE \\ &= AD(AD + DE) \\ &= \overline{AD}^2 + AD \times DE \\ &= \overline{AD}^2 + BD \times CD \quad \left[\begin{array}{l} \text{prod. of seg. of } AE \\ = \text{“ \ “ \ “ \ “ } BC \end{array}\right] \end{aligned}$$

EXERCISES

177. The sides of a triangle are 7, 8, and 12. The longest side of a similar triangle is 30. What are the other two sides?

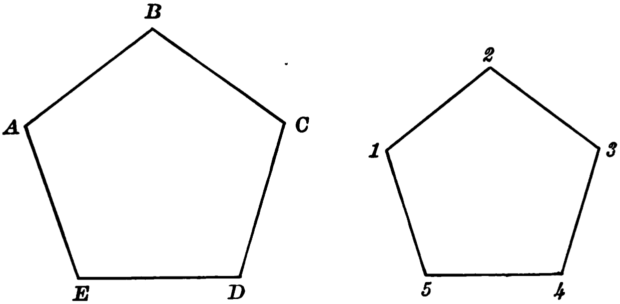
178. The bases of two similar triangles are 5 ft. 4 in. and 3 ft. 4 in. The altitude of the first is 4 ft. 10 in. What is the altitude of the second?

REGULAR POLYGONS — DEFINITION

224. A **Regular Polygon** is one that is both equiangular and equilateral.

PROPOSITION XXVI

225. Theorem. *Regular polygons of the same number of sides are similar.*



Dem. $AB = BC = CD$, etc. [sides of reg. ⬠]

$12 = 23 = 34$, etc. [" " " "]

$\frac{AB}{12} = \frac{BC}{23} = \frac{CD}{34}$, etc. [divide $=$ by $=$]

$A = 1$ [each $\angle = \frac{2n-4}{n}$ rt. $\angle$s]

⬠s are similar [mut. $=\angle$ and hom. s. prop.]

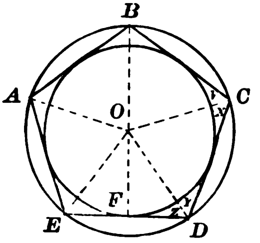
EXERCISES

179. In Prop. XIX, if AC, BC, 23, and 14 are respectively 48, 36, 27, and 25, what are 13 and AD?

180. Two isosceles triangles are similar if any angle of one equals the homologous angle of the other.

Proposition XXVII

226. Theorem. *A circle may be circumscribed about, or inscribed within, any regular polygon.*



Appl. *ABCDE* = reg. ⬠. Circles may be circumscribed and inscribed.

Cons. Take *O*, the centre of a ⊙ passing through three consecutive vertices *E*, *D*, and *C*. [determined by ⊥ bisectors of *ED* and *DC*]
Join *OB*, *OA*, etc.

Dem. 1. $X = Y$ [OCD = isos. △ (= radii)]

$\triangle$s $\left.\begin{array}{l} ODE \\ OCB \end{array}\right| =$ [2 s. and inc. ∠]

$OB = OE$

i.e. ⊙, which passes through *E*, *D*, and *C*, also passes through *B* and [similarly] *A*, etc., and a circle is circumscribed about the regular polygon.

2. *ED*, *DC*, etc., are = distant from centre *O* [chords of outer ⊙, = by hyp.]
⊙ described about *O* as a centre, with radius *OF* (⊥ to *ED*), will be an inscribed ⊙ [will touch sides of polygon]

227. Definitions. The **radius** of a regular polygon is the radius of the circumscribed circle, as *OE*.

The **apothem** of a regular polygon is the radius of the inscribed circle, as *OF*.

The **central angle** of a regular polygon is the angle between two consecutive radii, as *EOD*.

228. Cor. I. The central angle of a regular polygon is equal to four right angles divided by the number of sides.

229. Cor. II. The central angle of a regular polygon is the supplement of any angle of the polygon.

$$\left[\left(\frac{2n-4}{n}\right) \text{ rt. } \measuredangle\text{s} + \left(\frac{4}{n}\right) \text{ rt. } \measuredangle\text{s} = 2 \text{ rt. } \measuredangle\text{s}\right]$$

EXERCISES

Find x, the unknown term, in the following proportions :

181. $2 : 3 = 14 : x$.

182. $3 : 5\frac{1}{4} = x : 63$.

183. $9 : x = a : b$.

184. $9 : x = x : 25$.

185. Write in the form of a proportion $12x = 72$.

186. If $mn = pq$, write all the possible proportions.

187. Illustrate, by using numbers, the effect of alternation in a proportion. Of inversion, composition, division, the ratio between the sum of the antecedents and the sum of the consequents, etc.

188. In Prop. XXI, if *AB*, *BC*, and *CA* are respectively 16, 12, and 20, what are *AD*, *DC*, and *BD*?

189. In Prop. XXI, if $BA : BC = m : n$, what is the ratio of the perimeters of the triangle *M* and *N*?

190. In Prop. XXI, if the ratio between the angles *A* and *C* equals 3 : 7, how many degrees are there in each of the acute angles at *B*?

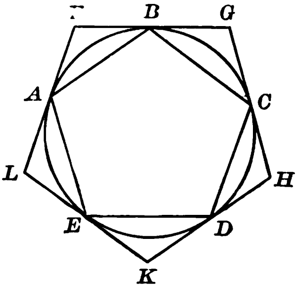
191. In Prop. XXIII, if $AE : ED = 8 : 5$ and $EB = 35$, what is *CE*?

192. Two sides of a triangle are 36 and 20 and the altitude upon the third side is 15. Find the radius of the circumscribed circle.

193. The three sides of a triangle are 10, 12, and 16. Find the bisector of the angle between the first two sides.

PROPOSITION XXVIII

230. Theorem. *Chords joining points of equal division on a circumference form a regular inscribed polygon, and tangents at these points form a regular circumscribed polygon.*



Appl. A, B, C, etc., are points of = division.

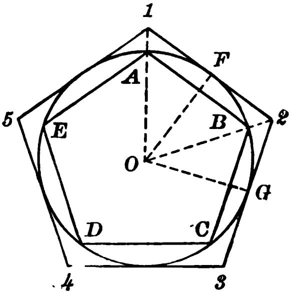
Prove $ABCDE$ = reg. ⬠, and that

Tangents at these points form $FGHKL$, a reg. ⬠

Dem.	1. $AB = BC = CD$, etc.	[chds. subtending = ⌒s]
	∠s $A = B = C$, etc.	[each ∠ meas. by one-half same no. of = ⌒s]
	$ABCDE$ = reg. ⬠	[equiangular and equilateral]
	2. △s BGC CHD etc. are = isos. △s	[s. and 2 adj. ∠s sides = (*Case* I) adj. ∠s = (meas. by one-half = ⌒s)]
	$G = H = K$, etc. $BG = GC = CH$, etc.	[hom. ∠s and sides]
	$FG = GH = HK$, etc.	[sums of = legs of = isos. △s]
	$FGHKL$ = reg. ⬠	[equiangular and equilateral]

PROPOSITION XXIX

231. Theorem. *A regular polygon inscribed in a circle being given, a similar polygon may be circumscribed about the circle.*



Appl. *ABCDE* = given reg. ⬠
A sim. ⬠ may be circumscribed
⬠ about the circle.

Cons. Bisect the ⌒s *AB*, *BC*, etc. at *F*, *G*, etc.
Draw tangents at *F*, *G*, etc.

Dem. *12345* = reg. ⬠ [*F* and *G* are points of = division, being midway points]

12345 is similar to *ABCDE* [reg. and has same number of sides]

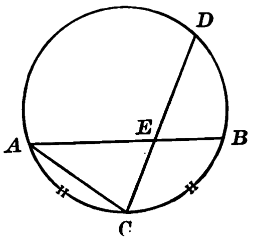
232. Cor. I. *OB* produced passes through *2* [*OB* = bisector of ∠*FOG* and the locus of all pts. = dist. from the sides. *2* is = dist. (= tang.)]

233. Cor. II. Chords *AF*, *BF*, *BG*, etc., form a regular inscribed polygon of double the number of sides of *ABCDE*.

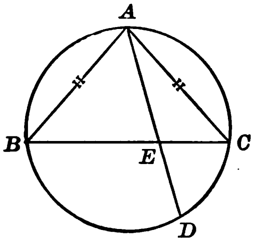
Tangents at *A*, *F*, *B*, *G*, etc., form a regular circumscribed polygon having double the number of sides of *12345*.

234. Cor. III. The sides of *ABCDE* and *12345* are parallel each to each.

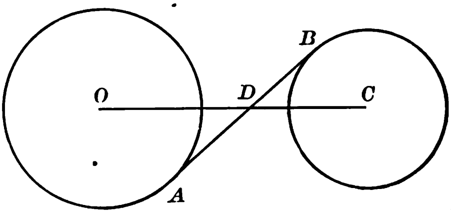
INTERPRET AND PROVE



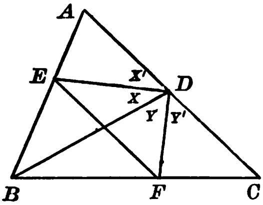
Ex. 194. Prove
$CE : AC = AC : CD$



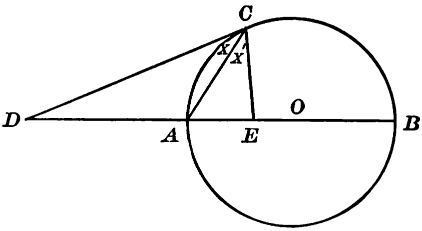
Ex. 195. Prove
$AD : AB = AB : AE$



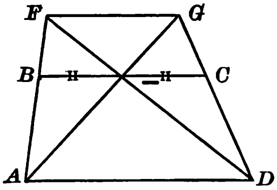
Ex. 196. $AB =$ common tangent.
Prove $AD : DB$ as the diameters.



Ex. 197. D bisects AC.
Prove $EF \parallel AC$.
State and prove the converse.

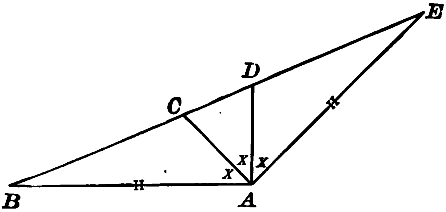


Ex. 198. C is any point.
Prove $AD : BD = AE : BE$

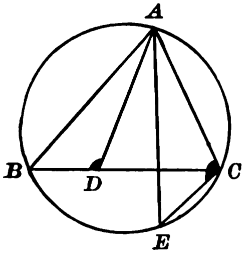


Ex. 199. Prove $FG \parallel AD$

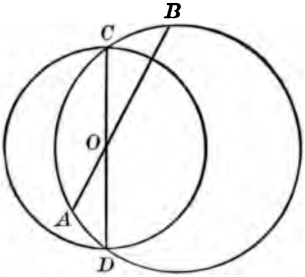
Interpret and Prove



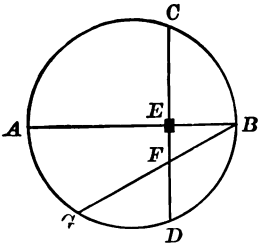
Ex. 200. $x = x = x =$ half a rt. ∠
Prove $BE : DE = DE : CD$



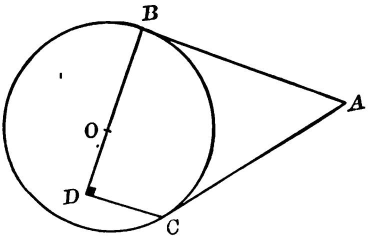
Ex. 201. Prove
$AB \times AC = AD \times AE$



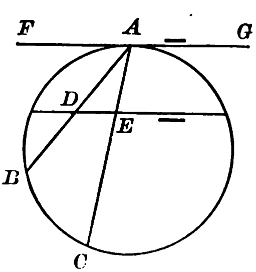
Ex. 202. Prove $AO \times BO =$ the square of the radius of the smaller circle.



Ex. 203. AB is a diam.
Prove $BA \times BE = BG \times BF$

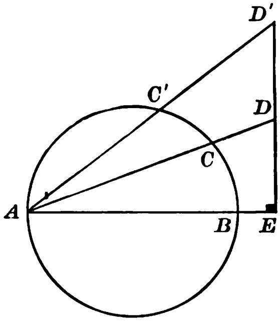


Ex. 204. AB and AC are tang.
Prove $CD : BD = OB : AB$

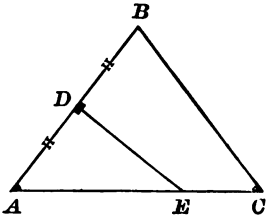


Ex. 205. $FG =$ tang. to ⊙
Prove $AB : AC = AE : AD$

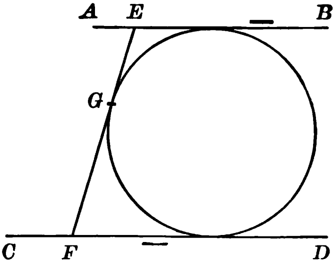
INTERPRET AND PROVE



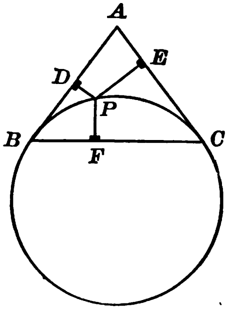
Ex. 206. AB = diameter produced to a fixed point E.
Prove $AC \times AD = AC' \times AD'$
i.e. $AC \times AD$ is constant.
Hint: join CB, and prove
$AC \times AD = AB \times AE$



Ex. 207. Prove
$AC : AB = AB : AE$



Ex. 208. AB, CD, and EF are tang. to ⊙. Prove radius is a mean proportional between GE and GF.



Ex. 209. AB and AC are tang.
Prove $PD : PF = PF : PE$
Hint: draw PB and PC.

EXERCISES

210. AD and BE are perpendiculars from the vertices of a triangle ABC to the opposite sides. Prove that

$$AC : DC = BC : EC.$$

211. ABC is a straight line and ABD and BCE are triangles on the same side of it, such that $AB : BE = BC : BD$ and the angle DBA equals the angle EBC. If AE and DC intersect at O, prove that AOC is an isosceles triangle.

212. A square is inscribed in a right triangle, having one of its sides coincident with a segment of the hypotenuse. Prove that a side of the square is a mean proportional between the other two segments of the hypotenuse.

213. The quadrilateral $ABCD$ is inscribed in a circle. CB and DA produced meet at E. A bisector of the angle E meets AB at F, and CD at G. Prove that

$$BA : BF = DC : DG.$$

214. If two circles touch each other, straight lines drawn through the point of tangency are cut proportionally by the circumferences.

215. Two secants are drawn to a circle from an outside point and their external segments are 20 and 24. If the internal segment of the former is 16, what is the internal segment of the latter?

216. Tangents to two intersecting circles from any point in their common chord produced are equal.

217. If two circles intersect, their common chord produced bisects their common tangents.

218. If, in a right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse, and circles inscribed in the two triangles formed, their radii are proportional to the legs of the right triangle.

219. ABC is a triangle inscribed in a circle, and D is the middle point of the arc AB. Prove that

$$\overline{CD}^2 = AC \times BC + \overline{AD}^2.$$

220. AB is a diameter of a circle, and tangents are drawn at A and B. A tangent to the circle at any other point C meets the two first tangents at E and F. O is the centre of the ⊙. Prove $\overline{EO}^2 : \overline{FO}^2 = CE : CF$.

221. ABC is an equilateral triangle inscribed in a circle. Lines are drawn to the three vertices from any point in the circumference, as, from D in the arc AC. Prove that $DB=DA+DC$. (Hint: draw chord AE parallel to DC.)

222. P is a point in AC, the diagonal of a parallelogram $ABCD$. EF is drawn through P, meeting BC and AD at E and F respectively, and GH is drawn through P, meeting AB and CD at G and H respectively. Prove EH parallel to GF.

223. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by a bisector of the exterior angle at the opposite vertex, minus the square of the bisector.

Hint. Circumscribe a circle and produce the diameter to the circumference as in § 223.

224. If the sides of a triangle are 8, 10, and 12, find the length of the bisector of the exterior angle formed by producing either of the first two sides (§ 202, and the preceding exercise).

225. A chord of a circle CD intersects a diameter AB at right angles, and E is any point in CD. AE and BE produced meet the circumference at F and G respectively. Prove that

$$CG : CF = DG : DF.$$

226. D is the middle point of the side BC of the triangle ABC. CE is any line intersecting AB at E and AD at F. Prove that the ratio of AF to FD is twice the ratio of $AE : EB$.

227. BC is the hypotenuse of a right triangle ABC, and AD a median. AE is drawn to the hypotenuse, making ABE an isosceles triangle. Prove that

$$BE : BA = BA : BD.$$

228. O is the centre of the circle inscribed in the triangle ABC and AO produced meets BC at D. Prove that

$$OA : OD = AB + AC : BC.$$

229. AB and CD intersect at E, and AC is parallel to DB. If F and G are points in AC and DB such that $FA : FC = GB : GD$, prove that F, E, and G lie in a straight line.

BOOK IV

AREAS — MEASUREMENT OF THE CIRCLE

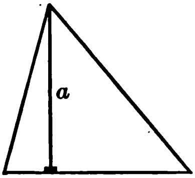
DEFINITIONS

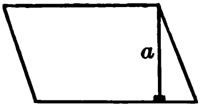
235. The **Area** of a polygon is the measure of its surface, *i.e.* the numerical value of its ratio to a **unit of surface,** arbitrarily taken as a standard. A unit of surface is usually a square whose side is some unit of length, as *a square foot,* or *a square decimeter.*

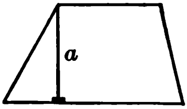
236. **Equivalent Polygons** are those which have equal areas.

237. The **Altitude** of a triangle, as previously stated, is the perpendicular distance from the opposite vertex to the base, or to the base produced, as a.

238. The **Altitude** of a parallelogram or a trapezoid is the perpendicular distance between the two bases, as a.



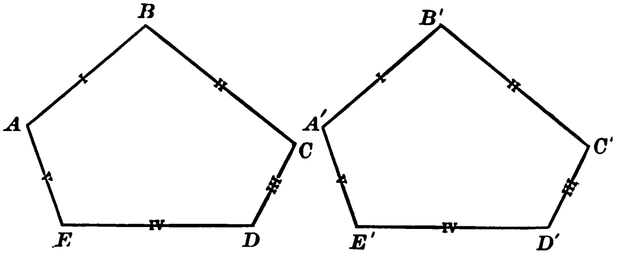




The words triangle, quadrilateral, etc., are hereafter often used meaning the *area of the triangle, area of the quadrilateral, etc.*

PROPOSITION I

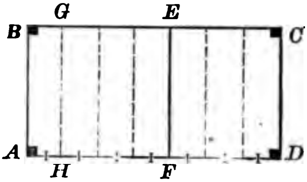
239. Theorem. *Two polygons mutually equiangular and mutually equilateral are equal.*



Dem. Prove by superposition, noting the coincidence

PROPOSITION II

240. Theorem. *Two rectangles having equal altitudes are to each other as their bases.*



Case I. When the bases are commensurable.

Appl. $\frac{ABCD}{ABEF} = \frac{AD}{AF}$, when AD and AF have a common measure

Cons. Apply com. meas., such as AH, to AD and AF, and let it be contained m times in AD, and n times in AF

Erect ⊥s GH, etc., at points of = division

Dem. $\frac{AD}{AF}=\frac{m}{n}$ [lines are to each other as their numerical measures]

Small rectangles $ABGH$, etc., are = and [mut. = ∠ and mut. = lat.]

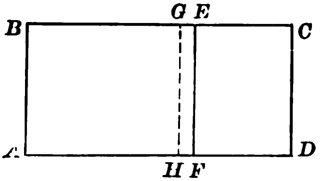
$ABGH$ is a *unit of measure* of the given rectangles

$\frac{ABCD}{ABEF}=\frac{m}{n}$ [rectangles are to each other as their numerical measure]

$\frac{ABCD}{ABEF}=\frac{AD}{AF}$ [Ax. 1]

Case II. When the bases are incommensurable.

Appl. $\frac{ABCD}{ABEF}=\frac{AD}{AF}$, even if AD and AF have no com. meas.



Cons. Apply some meas. of AD to AF.
Let it be contained a certain no. of times to the point H with a remainder HF, < measure

Dem.

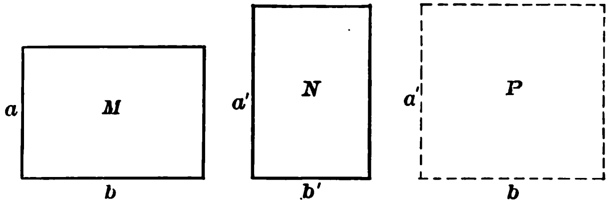
1.	AH = a variable, and	AF = its limit	as in Proposition XI, Bk. II, and Proposition X, Bk. III
2.	$ABGH =$ " "	$ABEF =$ " "	
3.	$\frac{AD}{AH} =$ " "	$\frac{AD}{AF} =$ " "	
4.	$\frac{ABCD}{ABGH} =$ " "	$\frac{ABCD}{ABEF} =$ " "	

$\frac{ABCD}{ABEF}=\frac{AD}{AF}$ [variables (3) and (4) = (*Case* I) ∴ limits (3) and (4) = (Theory of limits)]

241. Cor. Rectangles having equal bases are to each other as their altitudes. [Either side may be considered a base.]

PROPOSITION III

242. Theorem. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Appl. Prove

$$\frac{M}{N}=\frac{ab}{a'b'}$$

Cons. Construct rectangle P, as indicated

Dem. $\frac{M}{P}=\frac{a}{a'}$ [having = bases]

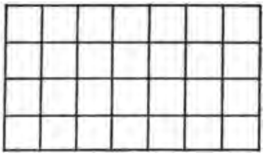
$\frac{P}{N}=\frac{b}{b'}$ [having = alt.]

$\frac{M}{N}=\frac{ab}{a'b'}$ [multiply]

PROPOSITION IV

243. Theorem. *The area of a rectangle equals the product of its base and altitude.*

[The area of the rectangle is to the square unit, taken as a standard, as the product of its base and altitude is to unity.]

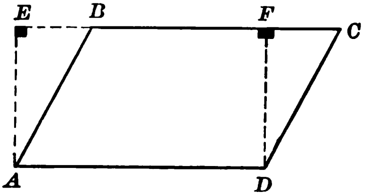


244. Cor. The area of a square is the square of one of its sides.

245. Sch. When the sides of the rectangle are multiples of the linear unit, the truth of the theorem is illustrated and made evident by the accompanying figure.

PROPOSITION V

246. Theorem. *The area of a parallelogram equals the product of its base and altitude.*



Appl. The area of $ABCD = AD \times DF$

Cons. Draw $AE \perp CB$ produced

Dem. $\begin{array}{r|}\triangle ABE \\ DFC\end{array} =$ $\left[\text{rt. } \triangle\text{s, hyp. and side}\right]$

$\square\, ABCD \equiv$ rectangle $AEFD$ $\left[\begin{array}{l}\text{add common part} \\ ABFD \text{ to each } \triangle\end{array}\right]$

$\square \equiv AD \times DF$ $\left[\begin{array}{l}\text{area of rectangle} \\ = AD \times DF.\end{array}\right]$

247. Cor. I. Parallelograms having equal bases and equal altitudes are equivalent.

248. Cor. II. Parallelograms having equal bases are to each other as their altitudes and those having equal altitudes are to each other as their bases.

249. Cor. III. Any parallelograms are to each other as the products of their bases and altitudes.

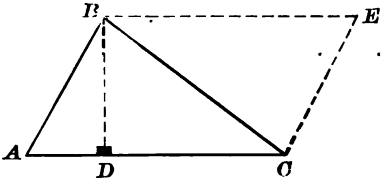
EXERCISES

230. If the base of a rectangle is 2 ft. 8 in., and its altitude 1 ft. 6 in., what is the side of an equivalent square?

231. The area of a rectangle is 2520 sq. in., and its altitude is $1\frac{1}{6}$ yd. What is its perimeter in feet?

Proposition VI

250. Theorem. *The area of a triangle equals one-half the product of its base and altitude.*



Appl. Area of $\triangle ABC = \frac{1}{2} AC \times BD$.

Cons. Draw $BE \parallel AC$
$CE \parallel AB$.

Dem. Area $ABC = \frac{1}{2} AC \times BD$ [$\triangle = \frac{1}{2} \square ABEC$]

251. Cor. I. Triangles having equal bases are to each other as their altitudes, and those having equal altitudes are as their bases.

252. Cor. II. Triangles are to each other as the products of their bases and altitudes (the factor $\frac{1}{2}$ disappears in the proportion).

EXERCISES

232. The base of a rectangle is 4 times its altitude, and its area is 21 sq. ft. 112 sq. in. What are its base and altitude?

233. If the altitude of a trapezoid is 20 in. and the bases 2 ft. 4 in. and 6 ft., respectively, what is its area?

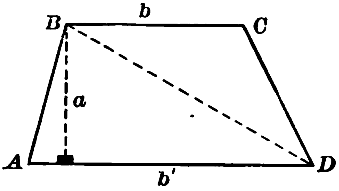
234. If the area of a trapezoid is 3136 sq. in., the altitude 2 ft. 8 in., and one base 6 ft. 8 in., what is the other base?

235. The diagonals of a parallelogram divide it into four equivalent triangles.

236. Lines drawn from two opposite vertices of a parallelogram to any point in the diagonal joining the other vertices, divide the parallelogram into two pairs of equivalent triangles.

PROPOSITION VII

253. Theorem. *The area of a trapezoid equals the product of its altitude and the half-sum of its bases.*



Appl. Area of trapezoid

$ABCD = a\left(\frac{b+b'}{2}\right)$

Cons. Draw BD.

Dem. Area $\triangle ABD = \frac{1}{2}ab'$

" $\triangle BCD = \frac{1}{2}ab$

" trapezoid $ABCD = a\left(\frac{b+b'}{2}\right)$

254. Cor. The area of a trapezoid is equal to the product of its altitude and its median.

[median equals one half the sum of the bases]

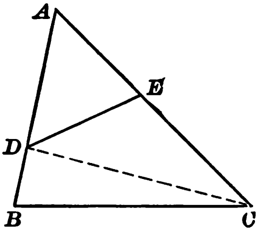
EXERCISES

237. The area of a rhombus is equal to one-half the product of its diagonals.

238. If the segments of the hypotenuse of a right triangle formed by a perpendicular from the vertex of the right angle are x and y, what is the area of the triangle? What is the area of each of the triangles formed by the perpendicular?

PROPOSITION VIII

255. Theorem. *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Appl. Prove $\frac{ABC}{ADE} = \frac{AB \times AC}{AD \times AE}$

Cons. Draw DC

Dem. $\frac{ABC}{ADC} = \frac{AB}{AD}$ [altitudes are equal, (bases in same st. l. and vertices at same pt.)]

$\frac{ADC}{ADE} = \frac{AC}{AE}$ ["]

$\frac{ABC}{ADE} = \frac{AB \times AC}{AD \times AE}$ [multiply]

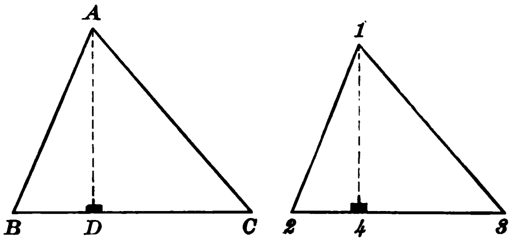
EXERCISES

239. The base and altitude of a triangle are 12 and 9 respectively. What is the altitude of a similar triangle whose base is 30?

240. Two triangles, having an angle of one supplementary to an angle of the other, are to each other as the products of the sides including those angles.

PROPOSITION IX

256. Theorem. *Two similar triangles are to each other as the squares of any two homologous sides.*



Appl. Similar $\triangle$

$\frac{ABC}{123}=\frac{\overline{AB}^2}{\overline{12}^2}$, etc.

Cons. Draw homol. altitudes AD and 14

Dem. $\frac{ABC}{123}=\frac{BC\times AD}{23\times 14}$ [$\triangle$ are as products of bases and altitudes]

or $\frac{ABC}{123}=\frac{BC}{23}\times\frac{AD}{14}$ [change form of right-hand member]

$\frac{ABC}{123}=\frac{BC}{23}\times\frac{BC}{23}=\frac{\overline{BC}^2}{\overline{23}^2}$ [subs. $\frac{BC}{23}$ for $\frac{AD}{14}$ hom. alt. of sim. $\triangle$ are as any two hom. sides]

$\frac{ABC}{123}=\frac{\overline{AB}^2}{\overline{12}^2}$, etc. [$\frac{AB}{12}=\frac{BC}{23}$, etc.]

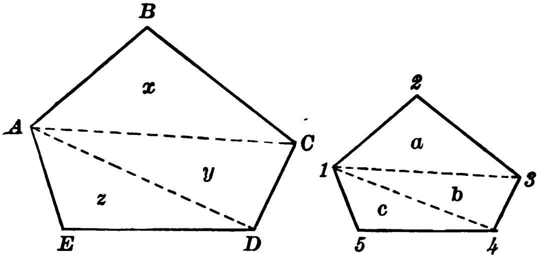
EXERCISES

241. The base of a triangle is m. What is the base of a similar triangle whose area is four times as much?

242. In the triangle ABC, $AC=2\,AB$, and BD is drawn to AC, making the $\angle ABD=\angle C$. Prove that the area of BDC is three times the area of BAD.

PROPOSITION X

257. Theorem. *Two similar polygons are to each other as the squares of any two homologous sides.*



Appl. Prove ⬠

$$\frac{A-E}{1-5}=\frac{\overline{AB}^2}{\overline{12}^2}, \text{ etc.}$$

Cons. Draw homol. diagonals

Dem. x sim. a, etc. [in sim. ⬠]

(1) $\dfrac{x}{a}=\dfrac{\overline{AB}^2}{\overline{12}^2}$ [sim. △]

$\dfrac{y}{b}=\dfrac{\overline{CD}^2}{\overline{34}^2}$ ["]

$\dfrac{z}{c}=\dfrac{\overline{ED}^2}{\overline{54}^2}$ ["]

$\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$ $\left[\dfrac{\overline{AB}^2}{\overline{12}^2}=\dfrac{\overline{CD}^2}{\overline{34}^2}, \text{ etc. } \left(\dfrac{AB}{12}=\dfrac{CD}{34}, \text{etc.}\right)\right]$

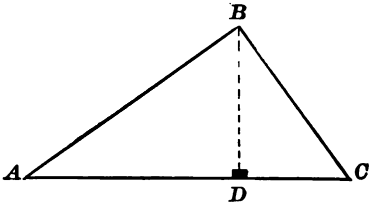
$\dfrac{x+y+z}{a+b+c}=\dfrac{x}{a}$ [sum. of ant. is to sum of conseq. as any ant. is to its conseq.]

$\dfrac{A-E}{1-5}=\dfrac{x}{a}=\dfrac{\overline{AB}^2}{\overline{12}^2}$, etc. [(1)]

Proposition XI

THE PYTHAGOREAN THEOREM

258. Theorem. *The square of the hypotenuse of a right triangle equals the sum of the squares of the legs.*



First Method

Appl. Prove $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$

Cons. Draw ⊥ BD

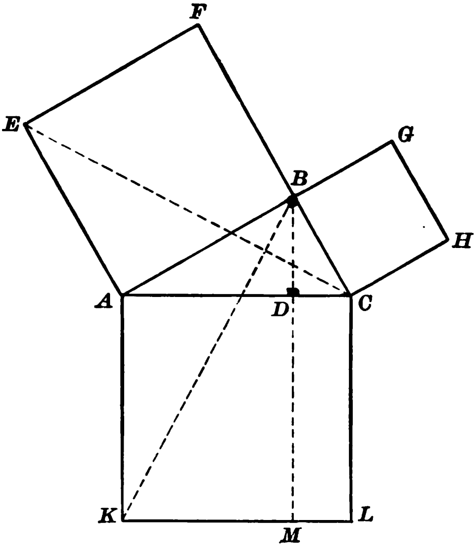
Dem. $\overline{AB}^2 = AC \times AD$ [either side about rt. ∠ = mean prop'l bet. hyp. and adj. segment]

$\overline{BC}^2 = AC \times DC$ ["]

$$\overline{AB}^2 + \overline{BC}^2 = AC(AD + DC)$$
$$= AC \times AC$$
$$= \overline{AC}^2$$

Note. — The discovery and statement of this proposition is commonly attributed to Pythagoras (about 500 B. C.). Some fifty or more demonstrations have been recorded, and the discussion of its relations has suggested scores of secondary exercises. The above demonstration is based upon similar triangles and is largely algebraic. The demonstration on the following page is geometric and constitutes Euclid's well-known "47th" (Euclid, Bk. I., Prop. 47, announced about 300 B. C.).

Second Method



259. Appl. Prove $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, or that the square described upon $AC \equiv$ the sum, etc.

Cons. Construct squares upon three sides of $\triangle$. Draw $BD \perp AC$ and produce to M. Draw BK and EC.

Dem.

$\triangle$s EAC | BAK =	[2 s. and inc. $\angle$ s. = (sides of square) ea. $\angle$ = rt. $\angle$ + CAB]
$EAC \equiv \frac{1}{2}$ sq. EB	[same base and alt. FBC = st. l. (sup. adj. $\angle$s at B)]
$BAK \equiv \frac{1}{2}$ rect. AM	[same base and alt.]
sq. $EB \equiv$ rect. AM	[doubles of equals]
" $BH \equiv$ " DL	[similarly]

" EB + sq. $BH \equiv$ sum of rect. AM and DL

sq. upon AB + sq. upon BC = sq. upon AC, or

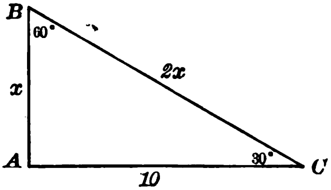
$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$

260. Cor. $\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2$

or $\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$ [transposing]

NOTE. — The theorem furnishes a method of computing any side a right triangle when the other two sides are known.

If only one side of a right triangle is known, the other two may found when the ratio, or some other relation, between them is known, in a 30° and 60° right triangle.



Let $AC = 10$, $B = 60°$, and $C = 30°$. Find AB and BC.

$BC = 2\,AB$ [§ 125]

Solving,
$$(2\,x)^2 - x^2 = \overline{10}^2$$
$$3\,x^2 = 100$$
$$x^2 = \tfrac{100}{3}$$
$$x = \tfrac{10}{3}\sqrt{3}$$
$$2\,x = \tfrac{20}{3}\sqrt{3}$$

It is convenient to know that certain integral numbers represent si of right triangles and that by taking multiples of these numbers, the si of similar triangles may be found. The following are such numbers:

3	4	5
5	12	13
7	24	25
8	15	17
9	40	41

EXERCISES

243. Find the shorter leg of a right triangle whose hypotenuse is and the other leg 56.

[Since 70 is 14 times 5, and 56 is 14 times 4, the shorter leg is times 3 (see note above).]

244. What is the hypotenuse of a right triangle whose other two sides are 264 and 495 feet? [Find the greatest common divisor of 264 and 495, for the multiple factor.]

245. A house is 28 feet wide and the ridge of the roof is 11 feet above the level of the eaves. Find the length of a rafter.

246. A ladder 60 feet long stands against a vertical wall. How far must it be moved from the wall at the bottom, to lower the top 5 feet?

247. The foot of a ladder 37 feet in length is placed 12 feet from the wall of a building. To what height on the wall will it reach?

248. A ladder 29 feet long just reaches a window 20 feet from the ground, and when turned about its foot it just reaches a window on the other side of the street, 18 feet from the ground. What is the width of the street?

249. A side of an equilateral triangle is 10. What is its altitude?

250. The altitude of an equilateral triangle is $6\sqrt{3}$. What is the length of a side?

251. What is the length of the side of a square whose diagonal is $72\sqrt{2}$?

252. One diagonal of a rhombus is three-fifths of the other, and the sum of the diagonals is 32. What is its area?

253. The area of a rhombus is 336 and a side is 25. Find its diagonals.

254. A circle has a diameter of 18 inches. What is the length of a tangent drawn from a point 32 inches outside the circumference?

255. The distance between the centres of two circles, whose radii are 4 and 11 feet respectively, is 25 feet. Find the length of the common tangents.

256. The area of an equilateral triangle is $25\sqrt{3}$. Find its side and its altitude.

257. The base of an isosceles triangle is 66, and each leg 65. Find its area.

258. The legs of an isosceles trapezoid are each 29 units long, and the bases are respectively 76 and 36. Find its area.

259. The legs of a trapezoid are 15 and 17, and the bases 21 and 29. Find its area.

260. The perpendicular from the vertex of the right angle to the hypotenuse of a right triangle divides the hypotenuse into two segments, $2\frac{2}{5}$ and $5\frac{2}{5}$ respectively. Find its area.

261. The hypotenuse of a right triangle is 29, and one leg is 20. Find its area.

262. Find the area of a square inscribed in a circle whose radius is 4. The area of the circumscribed square.

263. A house is 30 feet by 40 feet, and each end of the house is in the form of a right isosceles triangle resting upon a square. Find the entire exterior surface of the house.

264. The longest diagonal AD of an irregular hexagon $ABCDEF$ is 36, and perpendiculars from B, C, E, and F are, respectively, 15, 9, 20, and 24. If AB, CD, DE, and FA are, respectively, 17, 15, 29, and 25, what is the area of the hexagon?

265. A room is 15 by 20 feet, and 12 feet high. What is the distance from an upper corner to the opposite lower corner?

266. The inside dimensions of a box are as 5, 12, and 84. The longest rod that can be put into the box has a length of 51. What is the smallest dimension of the box?

267. The sides of two equilateral triangles are 3 feet and 4 feet respectively. Find the side and the area of the equilateral triangle equivalent to their sum?

268. The sum of the bases of a trapezoid is 300 yards, and its area is $3\frac{3}{4}$ acres. Find its altitude.

269. The base and altitude of a triangle are respectively 16 and 12 feet. Find the area of the trapezoid formed by a parallel to the base at a distance of 8 feet.

270. If the altitude of one equilateral triangle is equal to the side of another, what is the ratio of their areas?

271. Two homologous sides of two similar triangles are 12 and 13. Find the homologous side of a similar triangle equivalent to their difference.

272. In a trapezoid whose area is 48 sq. ft., one base is 6 ft. and the altitude is 4 ft. Find the length of a line drawn between the legs parallel to the base and 1 foot from it. 3 feet.

273. The sides of a right triangle are as 3 : 4 : 5. If the area is 150, find the altitude upon the hypotenuse.

274. The diagonals of a trapezoid are 15 and 18. If the segments of the smaller are 5 and 10, find the segments of the longer.

EXERCISES UPON THE PYTHAGOREAN THEOREM

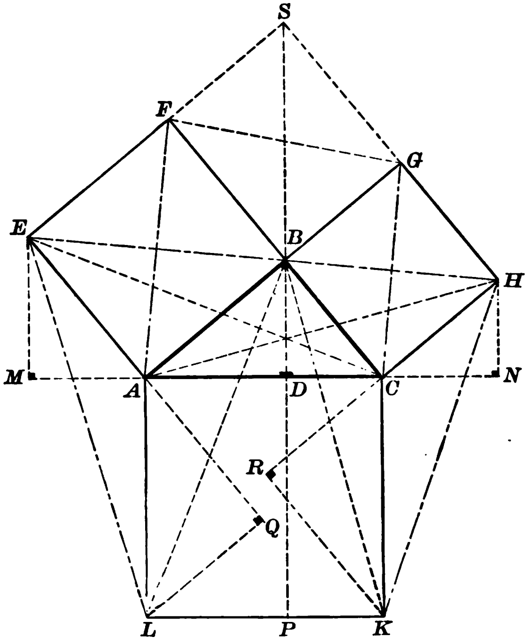
275. Prove FA parallel to GC.

276. That EBH is a straight line.

277. That $EM = AD$ and $HN = DC$.

278. That SBD is a straight line.

279. That in the figure as drawn there are six other triangles, each equal to ABC.



280. That $\overline{AG}^2 = \overline{AC}^2$ plus 4 times ABC.

281. That the sum of perpendiculars from F and G to $AC = AC + 2\,BD$.

282. That the sum of the angles AEL, ALE, CHK, and CKH equals a right angle.

283. That $\overline{BL}^2 - \overline{BK}^2 = \overline{BA}^2 - \overline{BC}^2$.

284. That $\overline{EL}^2 + \overline{HK}^2 = 5\,\overline{AC}^2$.

285. If LA produced meets ES at I and KC produced meets HS at J, prove that $AIJC$ is a square.

286. That $SIJ = ABC$.

287. If HR meets AQ at U, prove that $ESHU$ is a square.

288. That the sum of the squares on the sides of the polygon $LEFGHK$ equals eight times the square on AC.

289. That the polygon $LEFGHK \equiv 2$ sq. $AK + 4\,\triangle ABC \equiv 4\,\triangle LBK$.

290. That EC is perpendicular to BL.

EXERCISES

291. The parallelogram formed by joining the middle points of the consecutive sides of a quadrilateral is equivalent to one-half the quadrilateral.

292. The middle point of one leg of a trapezoid is connected with the extremities of the other leg. Prove that the largest triangle formed is equivalent to one-half the trapezoid.

293. Any straight line drawn through the point of intersection of the diagonals of a parallelogram divides the parallelogram into two equivalent figures.

294. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the altitude of the triangle.

295. The area of a triangle is equal to the product of the semi-perimeter and the radius of the inscribed circle.

296. If an angle of a triangle is 30°, prove that the area of the triangle equals one-fourth the product of the sides including the angle.

297. Prove that the area of a right isosceles triangle is equal to one-fourth the area of the square described upon its base.

298. $ABCD$ is a square. A line from A to the middle point of BC, and similar lines from the other vertices, form a square $\equiv \frac{1}{5} ABCD$.

299. $ABCD$ is a parallelogram, E the middle point of AB, and F the middle point of CD. CE and AF intersect BD at H and L. BF and ED intersect AC at K and G. Prove that $GHKL$ is a $\square \equiv \frac{1}{6} ABCD$.

300. If a quadrilateral is divided into two equivalent triangles by a diagonal, this diagonal is bisected by the other.

301. The bases of an isosceles trapezoid are as 2 : 1. If one diagonal is 18, find the segments of the other diagonal.

DEFINITIONS

261. The **projection of a point upon a line** is the foot of the perpendicular drawn from the point to the line, as D is the projection of A upon the line BC, in the figures below.

262. The **projection of a line upon a line** is that part of the second line that is between the feet of the perpendiculars drawn from the extremities of the first to the second, as the projection of the line AC upon the line BC is DC.

PROPOSITION XII

263. Theorem. *In any triangle the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of these sides and the projection of the other upon it.*

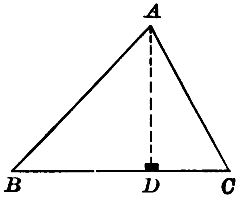


FIG. 1.

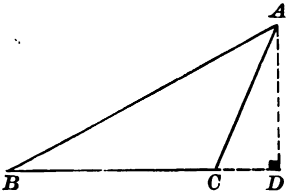


FIG. 2.

Appl. Prove $\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 - 2\,BC \times BD$

Cons. Draw $AD \perp BC$

Dem. (1) $DC = BC - BD$ (Fig. 1)

(2) $DC = BD - BC$ (Fig. 2)

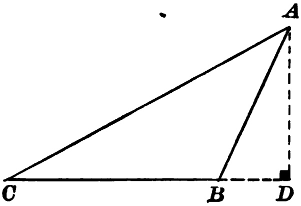
$$\overline{DC}^2 = \overline{BC}^2 + \overline{BD}^2 - 2\,BC \times BD \qquad \left[\begin{array}{l}\text{squaring (1)}\\ \text{or (2).}\end{array}\right]$$

$$\overline{AD}^2 + \overline{DC}^2 = \overline{BC}^2 + \overline{BD}^2 + \overline{AD}^2 - 2BC \times BD \qquad \left[\begin{array}{l}\text{add } \overline{AD}^2\\ \text{to each member}\end{array}\right]$$

$$\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 - 2\,BC \times BD \qquad \left[\begin{array}{l}\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2 \text{ (Pythag.)}\\ \overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2 \text{ (Pythag.)}\end{array}\right]$$

Proposition XIII

264. Theorem. *In an obtuse-angled triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides and the projection of the other upon it.*



Appl. Prove $\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2 + 2\,CB \times BD$

Cons. Draw $AD \perp CB$ (produced)

Dem. $CD = CB + BD$

$\overline{CD}^2 = \overline{CB}^2 + \overline{BD}^2 + 2\,CB \times BD$ [squaring]

$\overline{AD}^2 + \overline{CD}^2 = \overline{CB}^2 + \overline{BD}^2 + \overline{AD}^2 + 2\,CB \times BD$ [add $\overline{AD}^2$]

$\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2 + 2\,CB \times BD$ [$\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2$ (Pythag.) $\overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2$ (Pythag.)]

EXERCISES

302. Two sides of a triangle are 12 and 16 and the altitude upon the third side is 8. Find the third side.

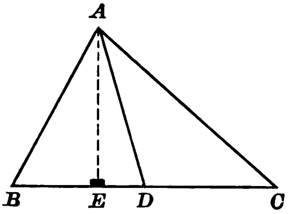
303. What kind of triangles — acute-angled, obtuse, or right — are those whose sides are 15, 17, 29? 7, 21, 25? 115, 35, 140? 8, 10, 12?

304. The square of the base of an isosceles triangle is equivalent to twice the product of a leg and the projection of the base upon the leg.

PROPOSITION XIV

265. Theorem. 1. *The sum of the squares of two sides of a triangle is equal to twice the square of half the third side plus twice the square of the median upon this side.*

2. *The difference of the squares of two sides of a triangle is equal to twice the third side multiplied by the projection of the median line upon it.*



Appl. Prove 1. $\overline{AC}^2 + \overline{AB}^2 = 2\,\overline{BD}^2 + 2\,\overline{AD}^2$

2. $\overline{AC}^2 - \overline{AB}^2 = 2\,BC \times ED$

Cons. Draw $AE \perp BC$

Dem. 3. $\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2 - 2\,BD \times ED$ [sq. of s. opp. acute ∠]

4. $\overline{AC}^2 = \overline{DC}^2 + \overline{AD}^2 + 2\,DC \times ED$ [sq. of s. opp. obtuse ∠]

$\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BD}^2 + 2\,\overline{AD}^2$ [add (3) and (4) $(BD = DC)$]

$\overline{AC}^2 - \overline{AB}^2 = 2\,(BD + DC) \times ED$ [subt. (3) from (4) $(BD = DC)$]

or $\overline{AC}^2 - \overline{AB}^2 = 2\,BC \times ED$

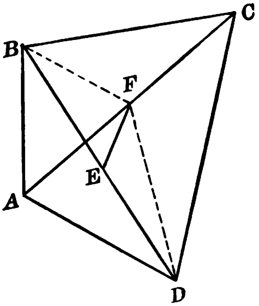
---

305. The base of a triangle is 100, and the other sides are 96 and 72. Find the median to the base.

306. Three times the sum of the squares of the sides of a triangle is equivalent to four times the sum of the squares of the medians of the triangle.

PROPOSITION XV

266. Theorem. *The sum of the squares of the four sides of a quadrilateral is equivalent to the sum of the squares of its diagonals plus four times the square of the line joining the middle points of the diagonals.*



Appl. Prove $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2$

Cons. Draw BF and DF

Dem.

$$\left.\begin{array}{l}\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AF}^2 + 2\,\overline{BF}^2 \\ \overline{CD}^2 + \overline{DA}^2 = 2\,\overline{AF}^2 + 2\,\overline{DF}^2\end{array}\right. \qquad \left[\begin{array}{l}\text{Prop. XIV} \\ \textit{Case } \text{I}\end{array}\right]$$

$$\begin{array}{l}\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 \\ \quad = 4\,\overline{AF}^2 + 2(\overline{BF}^2 + \overline{DF}^2)\end{array} \qquad \left[\text{add}\right]$$

$$= 4\,\overline{AF}^2 + 2(2\,\overline{BE}^2 + 2\,\overline{EF}^2) \qquad \left[\begin{array}{l}\text{substitute by} \\ \text{Prop. XIV, } \textit{Case } \text{I}\end{array}\right]$$

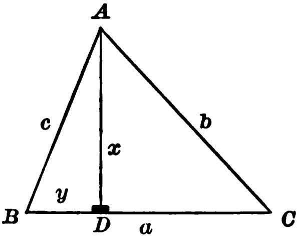
$$= 4\,\overline{AF}^2 + 4\,\overline{BE}^2 + 4\,\overline{EF}^2$$

$$= \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2 \qquad \left[\begin{array}{l}\text{sq. of } AC = 4 \text{ times} \\ \text{sq. of } AF\text{, etc.}\end{array}\right]$$

267. Cor. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals. [The line joining the middle points of the diagonals equals zero.]

FORMULÆ OF COMPUTATION

268. Problem. To compute the altitude and area of a triangle in terms of its sides.



Appl. Let a, b and $c =$ sides of $\triangle$ and $x =$ altitude

$$b^2 = a^2 + c^2 - 2ay \qquad \text{[sq. of side opp. acute } \angle\text{]}$$

$$y = \frac{a^2 + c^2 - b^2}{2a} \qquad \text{[transposing]}$$

$$x^2 = c^2 - y^2 \qquad \text{Pythag. theo.]}$$

$$= c^2 - \frac{(a^2 + c^2 - b^2)^2}{4a^2}$$

$$= \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4a^2}$$

$$= \frac{[2ac + (a^2 + c^2 - b^2)][2ac - (a^2 + c^2 - b^2)]}{4a^2}$$

$$= \frac{(a + b + c)(a - b + c)(b + c - a)(a + b - c)}{4a^2}$$

$$= \frac{[2s][2(s - b)][2(s - a)][2(s - c)]}{4a^2} \qquad \left[\text{let } s = \frac{a + b + c}{2}\right]$$

$$= \frac{4}{a^2}(s)(s - a)(s - b)(s - c)$$

$$x = \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)} \qquad (1)$$

$$\text{Area} = \frac{ax}{2} \qquad \left[\text{half base} \times \text{alt.}\right]$$

$$= \frac{a}{2} \cdot \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \qquad (2)$$

269. Problem. To compute the radius of a ⊙ circumscribed about a △ in terms of the sides of the △, a, b, and c.

Let $l =$ alt. of △ upon side a.

$$bc = 2R \times l \qquad \text{[see 221]}$$

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \qquad \left[\begin{array}{l}\text{subs. for } l \text{ its}\\ \text{value as found in}\\ \text{268 (1)}\end{array}\right]$$

270. Problem. To compute the area (S) of a triangle in terms of its sides and the radius of the circumscribed circle.

$$R = \frac{abc}{4S} \qquad [269]$$

whence

$$S = \frac{abc}{4R}$$

EXERCISES

307. Find the area of the triangle whose sides are 20, 34, and 42.

308. Find the area of the triangle whose sides are 21, 29, and 40.

309. If the sides of a triangle are a, b, and c, and the segments of the third side, formed by a bisector of the opposite angle, are x and y, prove that the area of one triangle equals $\frac{x}{x+y}$ times the area of the whole triangle. Prove that the area also equals $\frac{a}{a+b}$ times the total area. What expresses the area of the other triangle?

310. The sides of a triangle are 8, 10, and 12. Find the area of the two triangles formed by a bisector of the angle between the first two sides.

311. The sides of a triangle are as $3 : 4 : 5$, and the altitude upon the third side is 48. Find its area.

312. $ABCD$ is a trapezium; $AB = 18$, $BC = 21$, $CD = 16$, $DA = 25$, and $AC = 29$. Find its area.

271. Problem. To compute the bisectors of the angle of a triangle in terms of its sides, a, b, and c.

Let $d = AD$ bisector of $\angle A$, and x and y segments, BD and DC, of the side a.

$$bc = xy + d^2 \qquad \text{[see 223]}$$

$$(1) \qquad \therefore d^2 = bc - xy$$

$$\frac{x}{c} = \frac{y}{b} \qquad \left[\text{see 200}\right]$$

$$\frac{x}{c} = \frac{y}{b} = \frac{x+y}{b+c} = \frac{a}{b+c} \qquad \left[\text{composition, etc.}\right]$$

$$\therefore x = \frac{ac}{b+c} \text{ and } y = \frac{ab}{b+c}$$

Whence $$d^2 = bc - \frac{a^2bc}{(b+c)^2} \qquad \left[\text{subs. in (1)}\right]$$

(2) and $$d = \frac{2}{b+c}\sqrt{bcs(s-a)} \qquad \left[s = \frac{a+b+c}{2}\right]$$

EXERCISES

313. Two sides of a parallelogram are 98 and 112, and one of the diagonals is 128. Find the other.

314. The three sides of a triangle are 14, 16, and 20. Find the length of the median drawn to the third side.

315. If a, b, and c are the three sides of a triangle, develop a formula for the median to any side [see 265].

316. If a, b, and c are the three sides of a triangle, find the segments of the sides formed by the bisectors of the angles.

317. If $a =$ a side of an equilateral triangle, prove that its area $= \frac{a^2\sqrt{3}}{4}$.

318. The radius of a circle is 12 and the perimeter of a circumscribed polygon is 90. What is the area of the polygon?

319. If the sides of a triangle are 15, 17, and 20, find the radius of the inscribed circle [Ex. 295, page 137].

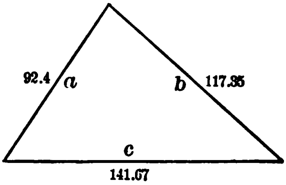
320. If the sides of a triangle are 17, 23, and 25, find the radius of the circumscribed circle.

321. The three sides of a triangle are 24, 30, and 40. Find the bisector of the angle between the first two sides.

Note. — The student should, before solving the following exercises, become familiar with the subject of Logarithms as given in the Appendix. The problems involving the use of the Metric System call for a ready knowledge of its various tables.

EXERCISE ILLUSTRATING THE USE OF LOGARITHMS

322. Find the area x of the triangle whose sides are respectively



$a = 92.4$
$b = 117.35$
$c = 141.67$

$$\begin{aligned} \operatorname{Log} x = \tfrac{1}{2}(\log s + \log(s-a) &+ \log(s-b) \\ &+ \log(s-c)) \end{aligned} \qquad \left[\begin{array}{l} 268 \text{ and Appendix,} \\ 11 \text{ to } 14 \end{array}\right]$$

$$\begin{array}{rrrr} 92.4 & & & \\ 117.35 & & & \\ 141.67 & 175.71 & 175.71 & 175.71 \\ 2\overline{)351.42} & \underline{92.4} & \underline{117.35} & \underline{141.67} \\ s = 175.71 & (s-a) = 83.31 & (s-b) = 58.36 & (s-c) = 34.04 \end{array}$$

$$\begin{array}{rl} \log 175.71 = & 2.24480 \\ \text{``} \quad 83.31 = & 1.92070 \\ \text{``} \quad 58.36 = & 1.76612 \\ \text{``} \quad 34.04 = & 1.53199 \\ & 2\overline{)7.46361} \\ \log x = & 3.73181 \\ x = & 5392.7 = \text{area.} \end{array}$$

EXERCISES INVOLVING THE USE OF LOGARITHMS

323. The first three terms of a proportion are 643.2, 961.45, and 75.438. Find the fourth term.

324. Find a third proportional to the numbers 33.47 and 56.83.

325. Find a mean proportional between 2748 and 5496.

326. Two sides of a triangle are .0436 and .0572. A parallel to the third side cuts off a distance .0291 from the first side. Find the segments of the second side.

327. The three sides of a right triangle are 252, 539, and 595. Find the segments of the hypotenuse formed by a perpendicular to it from the vertex of the right angle. Find also the length of the perpendicular.

328. A tangent and a secant from the same point to a circle are respectively 7.825 and 9.435. Find the internal segment of the latter.

329. Two sides of a triangle are respectively 7663^{mm} and 64.25^{dm}, and the altitude upon the third side is $.05872^{Hm}$. Find in meters the radius of the circumscribed circle.

330. The sides of a triangular field are 3052.6 meters, 6730.8 meters, and 4596 meters. Find its area in hectares.

331. *ABCD* is a trapezium. $AB = 192.9$ meters.
$BC = 296.2$ “
$CD = 1122.8$ “
$DA = 960.5$ “
$AC = 382.4$ “

Find its area in square kilometers.

332. The three sides of a triangle are 27.54^{dm}, 44.67^{dm}, and 56.92^{dm}. Find the three altitudes of the triangle.

333. Find its area in sq. meters.

334. Find the bisectors of its angles, and its medians.

335. Find the radius of the inscribed and of the circumscribed circles.

336. Two sides of a triangle are 1621.4 and 2112.8, and the angle between them is 30°. Find its area.

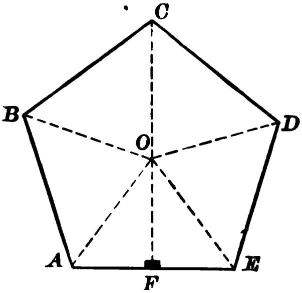
337. The radius of a circle is 324.6^{cm}. Find in sq. dm. the area of the inscribed equilateral triangle.

338. A sheet of copper is in the form of an isosceles trapezoid. The bases are 1^{m} and 25.6^{dm}, respectively, and the altitude is 875.5^{mm}. If the copper weighs .4 grams to the square centimeter, find the weight of the sheet in kilos.

339. A disc of iron is 7.8 times as heavy as water, measures 1^{dm} square, and weighs $.9^{Hg}$. If copper is 8.95 times as heavy as water, find the weight of a disc of the same thickness 1.643^{m} long and $.765^{m}$ wide.

PROPOSITION XVI

272. Theorem. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Appl. The area of $ABCDE = \frac{1}{2}pa$

Cons. Draw radii.
Draw OF (apothem)

Dem.

$$\left.\begin{array}{r}\triangle AOE\\ EOD\\ \text{etc.}\end{array}\right| = \qquad\qquad [3\text{ s.} = 3\text{ s.}]$$

$$OF = \text{alt. of all the } \triangle$$

$$\text{Area } AOE = \tfrac{1}{2}AE \times OF$$

$$\text{“} \quad EOD = \tfrac{1}{2}ED \times OF, \text{ etc.}$$

$$\text{“} \quad ⬠ = \tfrac{1}{2}(AE + ED + \text{etc.}) \times OF \qquad [⬠ \equiv \text{sum of } \triangle]$$

$$= \tfrac{1}{2}pa$$

EXERCISES

340. A field in the shape of a regular hexagon has a side equal to 125^{m}. Find its area in hectares.

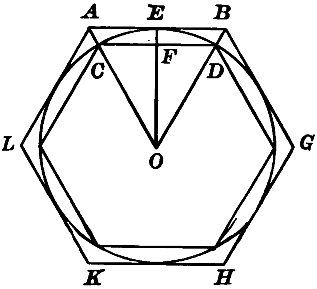
341. The area of a circumscribed equilateral triangle is how many times the area of the inscribed equilateral triangle?

342. The radius of an inscribed regular polygon is a mean proportional between its apothem and the radius of the similar circumscribed polygon.

PROPOSITION XVII

273. Theorem. *If the number of sides of an inscribed and of a similar circumscribed regular polygon is indefinitely increased,*

1. *The apothem of the inscribed polygon approaches the radius of the circle as a limit.*
2. *The perimeters of both polygons approach the circumference of the circle as a limit.*
3. *The polygons approach the circle as a limit.*



Appl. 1. Prove $CO =$ limit of FO.
2. Circumf. = " " both perimeters.
3. Area of ⊙ = " " areas of ⓢ.

Cons.

Dem. 1. $CO < CF + FO$ [st. l. < broken l.]
$CO - FO < CF$

$CO - FO$ approaches 0
or
$CO =$ limit of FO

[indef. increase no. of s.
length of each s. CD approaches 0
∴ CF ($\frac{1}{2}$ CD) approaches 0
∴ $CO - FO$ " 0]

2. Let P and p = perimeters of ⓢ.
A " a = apothems " "
C = circumf. of ⊙
$P : p = A : a$ [?]

$$\frac{P-p}{p}=\frac{A-a}{a}$$ [division]

$$P-p=\frac{p}{a}(A-a)$$

$P-p$ approaches 0 [indef. increase no. of s.
$A-a$ app's 0 (*Case* I)
$\therefore \frac{p}{a}(A-a)$ app's 0]

$P-C$ approaches 0 [C is between P and p]
$C-p$ " 0

or P and p approach C as a limit.

3. Let K and k = areas of ⓢ
and S = area of ⊙.

Dem. $$\frac{K}{k}=\frac{\overline{OA}^2}{\overline{OC}^2}=\frac{\overline{OA}^2}{\overline{OE}^2}$$ [sim. reg. ⓢ are as sq. of radii]

$$\frac{K-k}{k}=\frac{\overline{OA}^2-\overline{OE}^2}{\overline{OE}^2}$$ [division]

$$K-k=\frac{k}{\overline{OE}^2}(\overline{OA}^2-\overline{OE}^2)$$

$$=\frac{k}{\overline{OE}^2}\times\overline{AE}^2.$$ [Pythag. theo.]

$K-k$ approaches 0 [indef. increase no of s.
AB approaches 0
AE " 0
$\overline{AE}^2$ " 0
$\frac{k}{\overline{OE}^2}(\overline{AE}^2)$ " 0]

$K-S$ approaches 0 [S is between K and k]
$S-k$ " 0

or K and k approach S as a limit

PROPOSITION XVIII

274. Theorem. *Circumferences of circles are to each other as their radii, or as their diameters; and the circles as the squares of their radii, or as the squares of their diameters.*

Dem. $$\frac{P}{p} = \frac{R}{r}$$

$$\frac{K}{k} = \frac{R^2}{r^2}$$

[perim. of sim. reg. ⬠ are as radii, and areas as sq. of radii].

$$\frac{C}{c} = \frac{R}{r}$$

$$\frac{S}{s} = \frac{R^2}{r^2}$$

[$C =$ limit of P
$c =$ " " p
$S =$ " " K
$s =$ " " k
etc.]

275. Cor. I.

$$\frac{C}{c} = \frac{D}{d}$$

[let $D =$ diameter
$= 2 \times$ radius]

$$\frac{S}{s} = \frac{4\,R^2}{4\,r^2} = \frac{(2\,R)^2}{(2\,r)^2} = \frac{D^2}{d^2}$$

i.e. Circumferences of two circles are to each other as the diameters, and the areas as the squares of the diameters.

276. Cor. II.

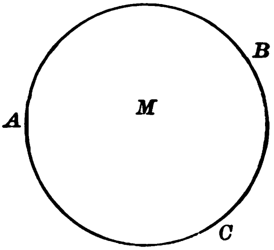
$$\frac{C}{D} = \frac{c}{d}$$

[Cor. I. by alternation]

i.e. The ratio between the circumference of a circle and its diameter is constant. This constant ratio is denoted by the symbol π, *i.e.* $\frac{C}{D} = \pi$.

PROPOSITION XIX

277. Theorem. *The area of a circle is equal to one-half the product of its circumference and its radius.*



Appl. Let M = a circle whose circumference = C, whose radius = R, and whose area = S.

Prove $M \equiv \frac{1}{2} C \times R$

Cons. Consider a regular polygon K inscribed in the ⊙

Dem.

$K \equiv \frac{1}{2} pa$	[½ perim. × alt.]
$M \equiv \frac{1}{2} CR$	[indef. increase no. of sides. K approaches M p " C a " R]

278. Cor. $C = 2\pi R = \pi D$ $\left[\frac{C}{D} = \pi\right]$

$$S = \frac{1}{2} \times 2\pi R \times R = \pi R^2$$

$$= \frac{1}{2}\pi D \times \frac{D}{2} = \frac{1}{4}\pi D^2$$

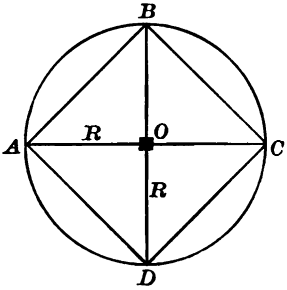
EXERCISES

343. The radius of a circle is 4. Find in terms of π the perimeter and the area.

344. The circumference of two concentric circles are, respectively, 210 and 280. Find in terms of π, the width of the ring formed by the two circumferences.

PROPOSITION XX

279. Theorem. *The extremities of two diameters which are perpendicular to each other determine the inscribed square.*



Appl. $AC =$ diam. $\perp BD$

Prove $ABCD$ is a square

Dem. $\overset{\frown}{s}\ AB = BC = CD = DA$ [= cent. ∠s (rt. ∠s)]

$ABCD =$ square [⊙ div'd into 4 = parts]

280. Cor. $\overline{AD}^2 = R^2 + R^2 = 2R^2$ [Pythag. theo.]

$AD = R\sqrt{2}$

i.e. The side of an inscribed square is equal to the radius of the circle multiplied by $\sqrt{2}$.

EXERCISES

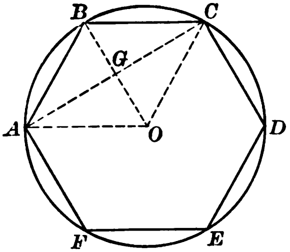
345. Find in terms of π the side of a square equivalent to a circle whose radius equals $4\sqrt{\pi}$.

346. Find the value of the apothem and area of a square inscribed in a circle whose radius is 4.

347. Find these values in the square circumscribed about the same circle.

PROPOSITION XXI

281. Theorem. *The radius of a circle equals the side of the regular inscribed hexagon.*



Appl. A chord AB, taken $=$ the radius AO, is the side of a regular inscribed hexagon

Cons. Join OA, OB

Dem. $ABO =$ equilateral $\triangle$ [sides = radii]

$\angle AOB = \frac{1}{3}$ of 2 rt. $\angle$s [$\angle$s =]

$= \frac{1}{6}$ of 4 rt. $\angle$s

Arc $AB = \frac{1}{6}$ circumf. [central $\angle = \frac{1}{6}$ (4 rt. $\angle$s)]

Chord $AB =$ side of hexagon

282. Cor. I. By joining alternate vertices, as A, C, and E, an equilateral triangle is formed.

283. Cor. II. Draw OC

AC bisects BO at rt. $\angle$s [$ABCO$ = rhombus]

$$OG = \tfrac{1}{2} R$$

$$\overline{AG}^2 = R^2 - \left(\frac{R}{2}\right)^2 = \frac{3R^2}{4}$$

$$AG = \tfrac{1}{2} R\sqrt{3}$$

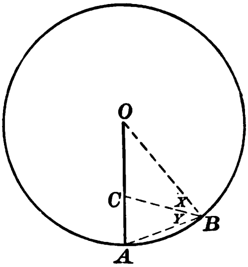
$$AC = R\sqrt{3}$$

i.e. The side of an inscribed equilateral triangle is equal to the radius of the circle multiplied by $\sqrt{3}$.

Definition. A line is divided into **extreme and mean ratio** when one segment is a mean proportional between the whole line and the other segment. If the line is divided internally, the longer segment is the mean proportional, and if the line is divided externally, the shorter segment is the mean proportional.

PROPOSITION XXII

284. Theorem. *If the radius of a circle is divided internally into extreme and mean ratio, the longer segment equals the side of the regular inscribed decagon.*



Appl. If OA is divided into extreme and mean ratio at C
$OC =$ side of a regular inscribed decagon

Cons. Measure $AB = OC$.
Join OB and CB

Dem.	$OA : AB = AB : CA$	[subs. AB for OC in ext. and mean ratio proportion]
	$\triangle$s OAB, CAB ∴ sim.	[∠ = ∠, and including s. prop'l preceding proportion, and ∠ A common]
	$Y = O$	
	$ABC =$ isos. △	[OAB is isos.]
	$CB = AB$	

$CO = CB$ [Ax. I. (both = AB)]

$X = O$ [isos. $\triangle OCB$]

$\angle B = 2\angle O$ [$B = X + Y$]

$A = 2\angle O$ [$A = B$]

$O = O$ [inserted for addition]

$O + A + B = 5\angle O$ [add]

or $\angle O = \frac{1}{5}(O + A + B)$

$= \frac{1}{5}(2 \text{ rt. } \angle s)$

$= \frac{1}{10}(4 \text{ rt. } \angle s)$

$\frown AB = \frac{1}{10}$ circumf. [$O = \frac{1}{10}(4 \text{ rt. } \angle s)$]

Chord $AB =$ side of regular inscribed decagon

285. Cor. I. By joining alternate vertices a regular inscribed pentagon is formed.

286. Cor. II. If radius of the circle $= R$ and $OC = x$

$$R : x = x : R - x$$

$$x^2 = R^2 - Rx$$

$$x^2 + Rx = R^2$$

whence $x = \frac{R}{2}(\sqrt{5} - 1)$ [solve the quadratic equation]

an expression for the side of a regular inscribed decagon in terms of the radius of the circle.

EXERCISES

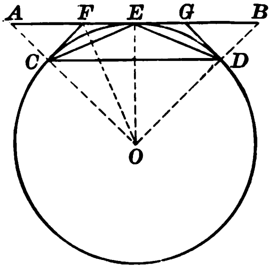
348. If the radius of a circle equals 4, find the area of the inscribed equilateral triangle.

349. Find the area of the equilateral triangle circumscribed about the circle whose radius is 6.

350. Find the apothem of the regular decagon inscribed in a circle whose radius is 3.

PROPOSITION XXIII

287. Problem. *Given the perimeters of a regular inscribed and of a similar circumscribed polygon, to find the perimeters of the regular inscribed and circumscribed polygons having double the number of sides.*



Appl. Let P and p denote the perimeters of the given polygons, and P' and p' the perimeters of the required polygons of double the number of sides. Find P' and p' in terms of P and p.

Cons. Let CD = side of inscribed ⬠

Thro' E, mid-pt. of $\frown CD$, draw tang. meeting OC and OD produced. Draw tangents at C and D. Connect OF, OE, CE, ED

If n = no. of sides of given polygons and $2n$ = no. of sides of required polygons,

Dem. $CD=\frac{p}{n}$; $AB=\frac{P}{n}$; $CE=\frac{p'}{2n}$ and $FG=\frac{P'}{2n}$

1. FO bisects $\angle AOE$ [tang. are drawn from F to ⊙]

$\frac{AF}{FE}=\frac{OA}{OE}$ [bisector of ∠ of △ divides opp. s., etc.]

$\frac{P}{p}=\frac{OA}{OE}$ (or OC) [perimeters are as radii]

$\frac{P}{p}=\frac{AF}{FE}$ [Ax. I.]

$$\frac{P+p}{p}=\frac{AF+FE}{FE} \qquad \left[\text{composition}\right]$$

$$\frac{P+p}{p}=\frac{AE}{FE}=\frac{\frac{1}{2}AB}{\frac{1}{2}FG}=\frac{\frac{P}{2n}}{\frac{P'}{4n}}=\frac{2P}{P'}$$

$$P'(P+p)=2P\times p \qquad [\text{clearing of fractions}]$$

$$P'=\frac{2P\times p}{P+p} \qquad (1)$$

2. $\left.\begin{matrix}CFE\\ CED\end{matrix}\right|$ sim. isos. $\triangle s$ $\qquad \left[\frac{\text{mut.} = \angle}{\text{alt. int. } \angle s =}\right]$

$$\frac{CD}{CE}=\frac{CE}{FE} \qquad \left[\text{hom. s. prop'l}\right]$$

$$\overline{CE}^2=CD\times FE$$

$$\frac{p'^2}{(2n)^2}=\frac{p}{n}\times\frac{P'}{4n}=\frac{P'p}{4n^2}$$

$$p'^2=P'\times p$$

$$p'=\sqrt{P'\times p} \qquad (2)$$

By the aid of formulae (1) and (2) P' can be found from P and p, and then p' from P' and p.

EXERCISES

351. In terms of r, the radius of the circle, find the apothem a, the side s, and the area k of the inscribed square.

352. Find a, s, and k for the circumscribed square.

353. Find a, s, and k for the inscribed and circumscribed regular hexagons.

354. The area of a regular inscribed hexagon is $24\sqrt{3}$. Find the diameter of the circle.

355. If the alternate vertices of a regular hexagon are joined, what is the ratio of the area of the hexagon, formed by these diagonals, to that of the given hexagon?

PROPOSITION XXIV

288. Problem. *To compute the approximate value of the ratio of the circumference of a circle to its diameter.*

Apply the formulae of the preceding theorem to the perimeters of the circumscribed and inscribed squares, finding thereby the perimeters of the circumscribed and inscribed octagons, as follows:

If diameter of circle $= 1$,

$$P = 4$$

$$p = 2\sqrt{2} = 2.82843$$

Substituting in formulae,

$$P' = \frac{2\,P \times p}{P + p} = 3.31371$$

$$p' = \sqrt{P' \times p} = 3.06147$$

Repeat the process, counting these last values as P and p, finding as before P' and p', the perimeters of circumscribed and inscribed polygons of sixteen sides.

Continuing the process the following table may be formed, in which it will be noticed that the values of the circumscribed and inscribed perimeters approach each other:

No. of sides	Perimeter of circumscribed polygon	Perimeter of inscribed polygon
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

The circumference of the circle, whose diameter is 1, is therefore greater than 3.14157 and less than 3.14163. That is, the approximate value of the ratio of the circumference of a circle to its diameter is 3.1416, correct to the fourth decimal place, or

$$\pi = 3.1416.$$

By more expeditious methods of trigonometry the value of this ratio has been computed to several hundred places.

This value 3.1416 is the ordinary approximation for practical problems and sometimes $3\frac{1}{7}$ is sufficient.

EXERCISES

356. The radius of a circle is 5 inches. What is its circumference? its area?

357. The diameter of a circle is 12. What is its circumference? its area?

358. The area of a circle is 176. What is its radius? its circumference?

359. The diameter of a wheel is $3\frac{1}{2}$ feet. How many revolutions would it make in traveling a mile?

360. What is the length of an arc of $10^\circ 15' 30''$ on a circumference whose radius equals 35 feet?

361. What is the area of a sector of 80° in a circle whose radius is 80 inches?

362. A wheel is of such a size that it turns twice in rolling 20 feet. What is its radius?

363. A wheel whose diameter is 8 feet makes 100 revolutions in a minute. How far does a point on the circumference travel in an hour?

364. If the equatorial diameter of the earth is 7924 miles, what is the length of 6 degrees of longitude at the equator?

365. The altitude of an equilateral triangle is 4 inches. Find the areas of the inscribed and the circumscribed circles.

366. The area of a circle is 432 square millimeters. What is its circumference in centimeters? How many degrees are there in an arc 1.6 centimeters in length?

367. If a rope 1 inch in diameter will support a ton, what will be the diameter of a similar rope to support 4 tons?

368. If a weight of 100 pounds is necessary to break a wire of $\frac{1}{16}$ inch diameter, what weight will be necessary to break a rod, of the same material, $\frac{1}{2}$ inch in diameter?

369. If a tank is filled in 1 hour and 4 minutes by a $\frac{1}{2}$-inch pipe, how long will it take a 4-inch pipe?

370. A 6-inch pipe will empty a reservoir in 3 hours. How long will it take if at the same time a 1-inch pipe is flowing in? a 3-inch pipe?

371. The side of a regular hexagon is 4 feet. Find the area of the circle inscribed in a regular hexagon 9 times as large as the given hexagon.

372. A circular garden plot 6 feet in diameter, is surrounded by a walk whose area equals that of the plot. Find its width.

373. The radius of a circle is 36 feet. Find the radius of the concentric circle which occupies one-fourth of it. Three-fourths of it.

374. Find the number of degrees in the arc of a circle, equal in length to the radius.

375. Find in terms of π the side of a square equivalent to a circle whose diameter is 10.

376. Find in terms of π the radius of a circle equivalent to a square whose perimeter equals 48.

377. The perimeter of a regular hexagon circumscribed about a circle is $24\sqrt{3}$. Find the circumference of the circle.

378. Find the area of a circle circumscribed about a regular hexagon whose area $= 32\sqrt{3}$.

379. Three equal circles, with a radius equal to 10, touch each other externally. Find the area enclosed by their convex arcs.

380. Two fields, one circular and one square, each contain an acre. What is the difference in their perimeters?

381. If the length of a quadrant is 10, express in terms of π the diameter and area of the circle.

382. Find the ratio of a circle to the altitude of the inscribed equilateral triangle.

383. The radii of two similar segments of circles are as 3 to 8. If the area of the first is 50, what is the area of the second?

384. Find the area of the segment of 60° in a circle whose diameter is 6.

385. If the radius of a circle equals 4, find the area of the segment whose arc is 30°.

386. Find the number of degrees in the arc of the sector whose area equals the square of the radius.

387. The radius of a circle is 42 inches, and two tangents from a point to its circumference are each $42\sqrt{3}$. Find the length of the two arcs determined by the points of contact.

388. Find the radius of a circle whose area is doubled by increasing its radius 3 feet.

389. The sum of the areas of a circle and an inscribed equilateral triangle is 12. Find the area of each.

If r represents the radius, a the apothem, s the side, and k the area, prove that

390. In a regular octagon,

$$s = r\sqrt{2-\sqrt{2}},\ a = \frac{r}{2}\sqrt{2+\sqrt{2}},\ \text{and } k = 2r^2\sqrt{2}.$$

391. In a regular dodecagon,

$$s = r\sqrt{2-\sqrt{3}},\ a = \frac{r}{2}\sqrt{2+\sqrt{3}},\ \text{and } k = 3r^2.$$

392. In a regular decagon,

$$s = \frac{r}{2}(\sqrt{5}-1)\ [\text{see } \S 286],\ a = \frac{r}{4}\sqrt{10+2\sqrt{5}},\ \text{and } k = \text{what?}$$

393. In a regular pentagon,

$$s = \frac{r}{2}\sqrt{10-2\sqrt{5}},\ a = \frac{r}{4}(1+\sqrt{5}),\ \text{and } k = \text{what?}$$

394. Prove that the square of the side of a regular inscribed pentagon, minus the square of the side of a regular inscribed decagon, equals the square of the radius.

395. The ring between two concentric circumferences is equivalent to a circle, whose diameter is a chord of the outer circle, tangent to the inner circle.

396. The area of a regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

397. The area of the square inscribed in a sector whose central angle is a right angle is equal to one-half the square of the radius.

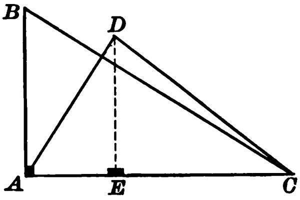
MAXIMA AND MINIMA

289. Definitions. Of magnitudes of the same kind, that which is greatest is a **maximum**; that which is least is a **minimum.**

290. Isoperimetric figures are those which have equal perimeters.

PROPOSITION XXV

291. Theorem. *Of all triangles formed with two given sides, that in which these sides are perpendicular to each other is a maximum.*



Appl. AC common to triangles ABC and ADC.

$AB = AD$. $BAC =$ rt. $\angle$.

Prove area $ABC >$ area ADC.

i.e. prove ABC a maximum.

Cons. Draw $\perp DE$

Dem. $DA > DE$ [?]

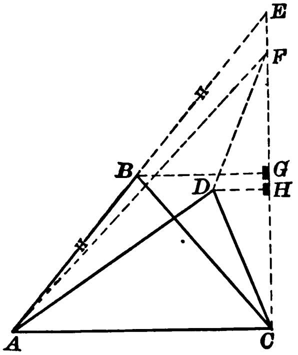
$BA > DE$

area $\triangle ABC >$ area $\triangle ADC$ [same base and greater alt.]

or ABC is a maximum.

PROPOSITION XXVI

292. Theorem. *Of isoperimetric triangles having the same base, the isosceles is the maximum.*



Appl. Let ABC = isos. △ and

ADC = any △ having same base and = perim.

Prove $ABC > ADC$, or that ABC is a maximum.

Cons. Produce AB to E. Take $DF = DC$, etc.

Dem. ACE = rt. △ [may be inscribed in semi ⊙ ($BA = BC = BE$)]

i.e. $EC \perp AC$

$AD + DF = AE$ [$AD + DC = AB + BE$ (cons.)]

$AD + DF > AF$ [?]

$AE > AF$

$EC > FC$ [greater oblique line cuts off greater dist. from foot of ⊥]

$GC = \frac{1}{2} EC$ and

$HC = \frac{1}{2} FC$ [⊥s to base in isos. △s]

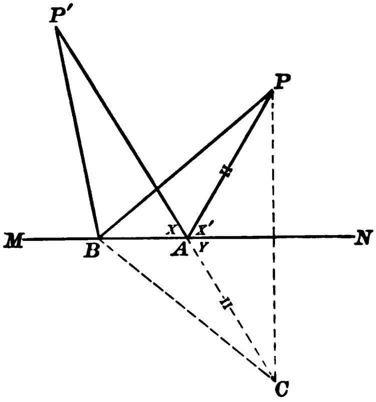
$GC > HC$ [$EC > FC$]

i.e. alt. of ABC > alt. of ADC.

∴ ABC is a maximum.

Proposition XXVII

293. Theorem. *The sum of two lines drawn from two points on the same side of a line to a given point of this line is a minimum when they make equal angles with the line.*



Appl. $X = X'$ $B =$ any point.

Prove $PA + P'A < PB + P'B$

Cons. Produce $P'A$ to C, making $AC = AP$. Join BC and PC.

Dem. $Y = X'$ [$Y = X$ (vert. $\angle$s)]

$AN = \perp$ bis. of PC [bis. of vert. $\angle$ of isos. $\triangle$]

$BC = BP$ [B is in $\perp$ bis.]

$P'C < PBC$ [?]

$P'A + PA < P'B + PB$ [subs. PA for CA and PB for CB]

i.e. $PA + P'A$ is a minimum.

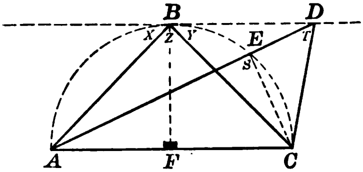
294. Sch. The theorem has several practical applications: A river flows in a straight course. P and P' are two points on the same side of the river and at unequal distances from it. If a man wishes to go by the shortest possible route from P

to P', taking water from the river on his way, he will direct his course so that the angle at which he approaches the river will equal the angle at which he leaves it.

Again, a ray of light striking a flat mirror obliquely leaves the mirror at the same angle as it first meets the mirror. The angle of *incidence* is said to be equal to the angle of *reflection.*

PROPOSITION XXVIII

295. Theorem. *Of triangles having the same base and altitude the isosceles has the minimum perimeter and the maximum angle at the vertex.*



Appl. ABC and ADC have same base AC and same alt. [alt. $=\perp$ dist. between $\parallel^s$ BD and AC]

Prove $AB+BC<AD+DC$, and prove $Z>T$

Cons. Circumscribe arc ABC. Join EC. Draw $BF\perp AC$

Dem. 1. $AB+BC<AD+DC$ [$X=Y$ (intercept $=$ $\frown$s)]

2. BF passes through centre [$\perp$ bis. of chord (isos. $\triangle$)]

$BF\perp BD$ [$\perp AC$ and $BD\parallel AC$]

$BD=$ tangent, and [$\perp$ at ext. of radius]

D is outside $\odot$ and AD cuts circumf. at E

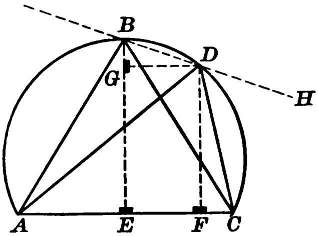
$Z=S$ [insc'd in same seg.]

$S>T$ [ext. $\angle$ of $\triangle$]

$Z>T$ [subs. Z for S]

PROPOSITION XXIX

296. Theorem. *Of triangles having equal bases and equal angles at the vertex, the isosceles has the maximum area and the maximum perimeter.*



Appl. ABC and ADC have same base AC and $= \angle$s at vertex [(insc'd. in same seg.)]

Prove ABC has greater area than ADC and prove $AB + BC > AD + DC$

Cons. Draw line through B and D

Draw altitudes of $\triangle$s and $DG \parallel AC$

Dem. BE produced (diam.) $>$ DF produced (any other chord)

$BE > DF$ [?]

$\triangle ABC > \triangle ADC$ [same base and greater alt.]

$\angle BDA = \angle HDC$ [meas. by $\frac{1}{2} = \frown$s]

$AD + DC =$ a minimum [make $= \angle$s with BH]

i.e. $AD + DC < AB + BC$ [since D is any point except B]

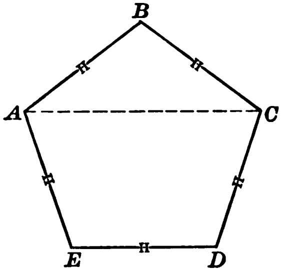
or $AB + BC$ is a maximum

---

398. The extremities of a sliding line AB touch two perpendiculars. In what position is AB when the triangle formed by the two lines is a maximum? If $AB = 10$, what is that area?

PROPOSITION XXX

297. Theorem. *Of isoperimetric polygons of the same number of sides, the maximum is equilateral.*



Appl. *ABCDE* = equilateral ⬠

Prove ⬠ > an isoperimetric ⬠ with unequal sides

INDIRECT PROOF

Dem. If *ABCDE*, a maximum, is not equilateral, in place of a triangle formed as *ABC* may be substituted an isosceles △ greater than *ABC* and having same perimeter [292].

∴ each two consecutive sides are equal, or the maximum polygon is equilateral.

EXERCISES

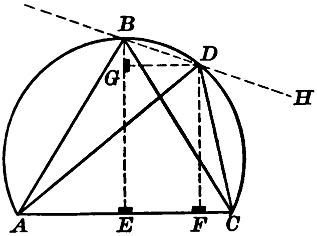
399. The angle between two lines drawn from any point in a straight line to the extremity of another line is a maximum when the point is the point of contact of a circumference passing through the extremities of the second line.

400. The sum of the squares of the two segments of a line is a minimum when the segments are equal.

401. The rectangle contained by the two segments of a line is a maximum when the segments are equal.

PROPOSITION XXIX

296. Theorem. *Of triangles having equal bases and equal angles at the vertex, the isosceles has the maximum area and the maximum perimeter.*



Appl. ABC and ADC have same base AC and $=\angle$s at vertex [(insc'd. in same seg.)]

Prove ABC has greater area than ADC and prove $AB+BC>AD+DC$

Cons. Draw line through B and D

Draw altitudes of $\triangle$s and $DG \parallel AC$

Dem. BE produced (diam.) $>$ DF produced (any other chord)

$BE>DF$ [?]

$\triangle ABC > \triangle ADC$ [same base and greater alt.]

$\angle BDA = \angle HDC$ [meas. by $\frac{1}{2}=$ ⌢s]

$AD+DC=$ a minimum [make $=\angle$s with BH]

i.e. $AD+DC<AB+BC$ [since D is any point except B]

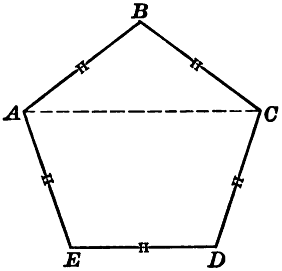
or $AB+BC$ is a maximum

---

398. The extremities of a sliding line AB touch two perpendiculars. In what position is AB when the triangle formed by the two lines is a maximum? If $AB=10$, what is that area?

PROPOSITION XXX

297. Theorem. *Of isoperimetric polygons of the same number of sides, the maximum is equilateral.*



Appl. $ABCDE =$ equilateral ⬠

Prove ⬠ $>$ an isoperimetric ⬠ with unequal sides

INDIRECT PROOF

Dem. If $ABCDE$, a maximum, is not equilateral, in place of a triangle formed as ABC may be substituted an isosceles △ greater than ABC and having same perimeter [292].

∴ each two consecutive sides are equal, or the maximum polygon is equilateral.

EXERCISES

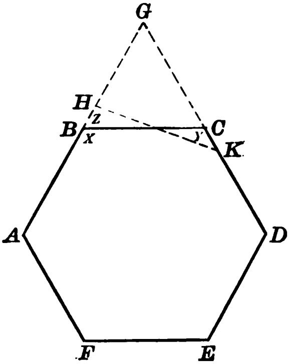
399. The angle between two lines drawn from any point in a straight line to the extremity of another line is a maximum when the point is the point of contact of a circumference passing through the extremities of the second line.

400. The sum of the squares of the two segments of a line is a minimum when the segments are equal.

401. The rectangle contained by the two segments of a line is a maximum when the segments are equal.

PROPOSITION XXXI

298. Theorem. *Of equilateral isoperimetric polygons of the same number of sides, the maximum is equiangular.*



Appl. Let $ABCDEF =$ a maximum of equilat. isoper. polygons; prove that $ABCDEF$ is equiangular.

Cons. Produce AB and DC to meet at G.

Suppose (indirect proof) $X > Y$

$Z < V$ [suppl. of X and Y]

$GB > GC$ [greater s. opp. greater ∠]

Measure $GH = GC$, and $GK = GB$. Draw HK

Dem. △ $BGC =$ △ HGK [2 s. and inc. ∠]

⬠ $ABCDEF \equiv$ ⬠ $AHKDEF$ [subtract △s from whole figure]

$HK = BC$ [= △s]

$BH = CK$ [subt. = from =]

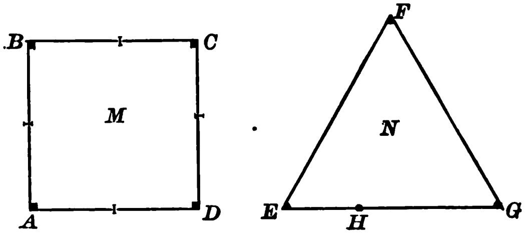
Perim. of given ⬠ = perim. of constructed ⬠ [addition of sides]

⬡ *AHKDEF* = max. of isoper. Ⓢ [≡ *ABCDEF*]
⬡ *AHKDEF* is equilat. [being a max.]
This is impossible [$AH > KD$]
∴ X is *not* $> Y$
X is not $< Y$ [similarly]
∴ $X = Y$, and ⬡ is equiangular.

299. Cor. Of isoperimetric polygons of the same number of sides the maximum is both equilateral and equiangular; that is, the maximum is regular.

PROPOSITION XXXII

300. Theorem. *Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum.*



Appl. M and N have equal perimeters.
Prove $M > N$

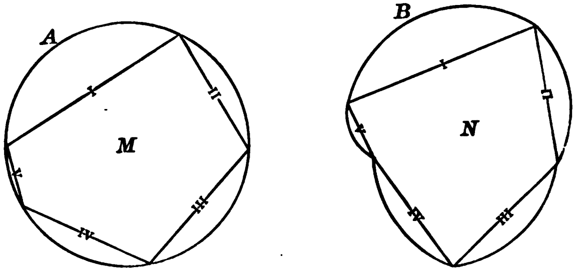
Cons. Consider N an irregular ⬡ of four sides, H being a vertex, and the $\angle EHG = 2$ rt. ∠s

Dem. $M > N$ [of isoper. Ⓢ the reg. is max.]
A ⬡ of five sides $> M$, and so on [similarly]

301. Cor. The circle is the maximum of isoperimetric plane figures, for as the number of sides of a regular inscribed polygon is indefinitely increased, the circle is the limit of the polygon.

PROPOSITION XXXIII

302. Theorem. *Of mutually equilateral polygons, the one which can be inscribed in a circle is a maximum.*



Appl. M and N are mutually equilateral.

M is inscribed in a ⊙, and N cannot be inscribed in a ⊙

Prove $M > N$

Cons. On sides of N describe arcs = respectively to the arcs on M

Dem. Perim. of A = perim. of B [sums of = $\frown$s]

$A > B$ [⊙ = max. of isoper. fig.]

$M > N$ [subtract = seg. from A and B]

EXERCISES

402. Of triangles inscribed in a circle the equilateral has the maximum perimeter.

403. The maximum rectangle that can be circumscribed about a given quadrilateral is equivalent to twice the quadrilateral.

404. Of equivalent regular polygons, that which has the maximum number of sides has the minimum perimeter.

405. The perimeter of an isosceles triangle is greater than that of an equivalent rectangle of the same altitude.

SYMMETRY

DEFINITIONS

303. Two points are **symmetrical** with respect to a third when the latter point bisects the straight line joining the given points.

Such a bisecting point is a **centre of symmetry.**

304. Two points are symmetrical with respect to a straight line, when the line bisects at right angles the straight line joining the given points.

Such a bisecting line is an **axis of symmetry.**

305. Two *figures* are symmetrical with respect to a centre, or to an axis, when their homologous vertices or other determining points are symmetrical with respect to the centre, or the axis.

306. A figure is symmetrical with respect to a centre if every straight line which passes through the centre cuts the figure in symmetrical points.

307. A figure is symmetrical with respect to an axis when the axis divides the figure into two figures, symmetrical with respect to that axis.

EXERCISES

Determine which of the following figures are symmetrical, both with respect to centres and with respect to axes:

406. A circle.
407. A square.
408. A rhombus.
409. A regular pentagon.
410. A regular hexagon.
411. The common part of two intersecting circles.
412. A parallelogram.
413. An isosceles trapezoid.
414. An equilateral triangle.
415. A sector of a circle.
416. A segment of a circle.

TOPICAL SUMMARY

308. Straight lines are equal if they

1. are homologous sides of equal triangles.
2. are legs of an isosceles triangle.
3. are opposite sides of a parallelogram.
4. in equal circles, are radii or diameters.
5. in equal circles, subtend equal arcs.
6. in equal circles, are equally distant from the centre.

309. Angles are equal if they are

1. complements, or supplements, of equal angles.
2. alternate-interior, or corresponding, angles.
3. homologous angles of equal triangles.
4. base angles of an isosceles triangle.
5. measured by equal arcs, in equal circles.

310. Lines are parallel if

1. alternate-interior (etc.) angles are equal.
2. perpendicular or parallel to the same line.

311. Triangles (any) are equal if they have

1. two sides and the included angle respectively equal.
2. a side and two angles respectively equal.
3. three sides respectively equal.

312. Right-triangles are equal if they have

1. the hypotenuse and a side respectively equal.
2. the hypotenuse and an acute angle respectively equal.
3. a side and an acute angle respectively equal.

NOTE. — The summary here given comprises the most important and most frequently used theorems of Plane Geometry. The list should practically be memorized, and employed in the solution of original exercises. The hypotheses of such exercises, when transferred to graphic figures, will in a large majority of instances suggest a solution along the lines of proof given in this summary.

313. Triangles are similar if

1. they are mutually equiangular.
2. their homologous sides are proportional.
3. an angle of one equals an angle of the other, and their including sides are proportional.

314. A proportion is established

1. by similar triangles (usually because they are mutually equiangular).
2. by a line parallel to a side of a triangle.
3. by a bisector of an angle of a triangle.
4. in a right triangle, by a perpendicular to the hypotenuse from the vertex of the right angle.
5. by intersecting chords.
6. by a tangent and a secant from a point to a circle.

315. Triangles (or parallelograms) are equivalent if

1. they have equal bases and equal altitudes.

316. Theorems involving the Theory of Limits:

1. In equal circles, the ratio between central angles (174).
2. A parallel to a side of a triangle (197).
3. Two rectangles having equal altitudes (240).
4. The circle the limit of regular polygons (273).

317. Important computation theorems:

1. The sum of the angles of a triangle (81).
2. Each angle of a polygon, $(2n-4)$ right angles (111).
3. The medians of a triangle (121).
4. A 30° and 60° right-triangle (125).
5. The measures of inscribed angles, etc. (176–182).
6. Algebraic theorems of proportion (183–195).
7. Segments of a side of a triangle formed by bisector of the opposite angle (200, 202).

8. In similar figures
    - *a.* lines (homologous) are as lines (209–214).
    - *b.* areas are as squares of lines (256, 257).
    - *c.* volumes are as cubes of lines (proved in Solid Geometry).

9. A perpendicular from the vertex of the right angle to the hypotenuse of a right triangle (215).

10. The diameter of a circle circumscribed about a triangle (221, 269).

11. The radius of a circle inscribed within a triangle (Ex. 295, p. 137).

12. The bisector of an angle of a triangle (223, 271).

13. Areas of rectangles, triangles, etc. (243–255).

14. The Pythagorean Theorem (258).

15. The squares of sides of a triangle opposite oblique angles (263, 264).

16. A median in a triangle (265).

17. The altitude and area of a triangle in terms of the semiperimeter (268).

18. The development of π (214, 274, 275, 276, 287, 288).

19. The area of a regular polygon (272); of a circle (277).

20. In a circle,

$$C = \pi D = 2\pi r,$$

$$S = \tfrac{1}{4}\pi D^2 = \pi r^2.$$

318. Other important or typical theorems, not readily classified in the preceding groups, are discussed in Sections 58, 93, 95, 101, 103, 113, 115, 117, 123, 148, 157, 161, 226, 230, 274, 281, 284, 287, 288, 299.

MISCELLANEOUS EXERCISES

417. AB is a perpendicular between two parallels, and AC an oblique line. CD is a second oblique line intersecting AB at E, and such that $ED = 2AC$. Prove that the angle ACB is three times the angle DCB.

418. *ABCD* is a parallelogram, and *E* the middle poi
and *DE* are produced to meet at *F*. Prove that △ *BFD*

419. Regular polygons of how many sides may be
pavement, if the blocks are all of one shape? of two s
shapes?

420. An inscribed trapezoid is isosceles.

421. An inscribed parallelogram is a rectangle.

422. Tangents at the vertices of an inscribed rectangle form a rhombus.

423. The line joining the middle points of the diagonals of a trapezoid is equal to one-half the difference of the bases.

424. The square on the sum *AC* of two lines *AB* and *BC* (Fig. 1) is equivalent to the sum of the squares on the two lines, plus twice the rectangle, contained by the lines.

Or, expressed algebraically, if $AB = a$, and $BC = b$,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

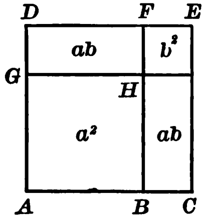


Fig. 1.

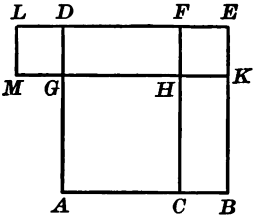
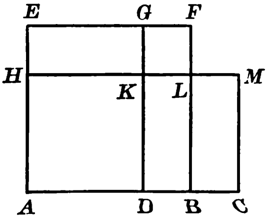


Fig. 2.

425. The square on the difference *AC* of two lines *AB* and *BC* (Fig. 2) is equivalent to the sum of the squares on *AB* and *BC*, minus twice the rectangle contained by *AB* and *BC*.

Or, $$(a - b)^2 = a^2 - 2ab + b^2.$$



426. The rectangle contained by the sum *AC* and the difference *AD* of two lines, *AB* and *BC*, is equivalent to the difference of their squares.

Or, $(a + b)(a - b) = a^2 - b^2$.

427. If a straight line is divided into two equal and also two unequal parts, the rectangle contained by the unequal parts plus

the square on the line between the points of division is equivalent to the square on half the line.

428. AB is the diameter of a circle, and AC, BD are chords intersecting within the circle at E. Prove $\overline{AB}^2 = AC \times AE + BD \times BE$ (§ 264, § 220).

429. C and D are the middle points of a chord AB and its subtended arc. If $BD = 4\sqrt{3}$, and $CD = 6$, find the diameter of the circle.

430. BC is a leg of a right-triangle ABC, right-angled at C, and D its middle point. Prove that $\overline{AB}^2 - \overline{AD}^2 = 3\,\overline{CD}^2$.

431. AB is the hypotenuse of a right-triangle ABC, and D its middle point. Prove that $\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2 = 8\,\overline{CD}^2$.

432. AC is the base of an isosceles triangle ABC, and AD is a perpendicular from A to BC. Prove that

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2 = 3\,\overline{AD}^2 + 2\,\overline{BD}^2 + \overline{CD}^2.$$

433. BC is a leg of a right-triangle, right-angled at B, and D any point in BC. Prove that $\overline{AC}^2 + \overline{BD}^2 = \overline{AD}^2 + \overline{BC}^2$.

434. In the triangle ABC, D is the middle point of BC, and E the middle point of AC. Any two parallels are drawn through D and E, meeting AB, or AB produced, at F and G. Prove that $DEFG$ is a ▱, $\equiv \frac{1}{2} \triangle ABC$.

435. Two parallel chords on opposite sides of the centre of a circle are 30 and 48 inches long, respectively, and the distance between their middle points is 27. What is the radius of the circle?

436. The radius of a circle is 12 inches, and one segment of a chord which passes through a point 8 inches from the centre of the circle is 5. What is the other segment?

437. The lateral surface of a cylindrical lead pencil is 7 square inches. What is the surface of a similar pencil twice as long and three times as thick?

438. DE is a parallel to the hypotenuse AB of a right-triangle, meeting AC at D and BC at E. Prove that

$$\overline{AB}^2 - \overline{AE}^2 = \overline{BD}^2 - \overline{DE}^2.$$

439. What fractional part of a triangle is between one of the sides and a parallel to this side drawn through the centre of gravity?

440. The sum of the squares of the lines drawn from any point on the circumference of a circle to the vertices of an inscribed square is equal to twice the square of the diameter of the circle.

441. If a, b, and c are three sides of a triangle, show t the two triangles, formed by a bisector of the angle be a and b, are respectively $\frac{a}{a+b}$, and $\frac{b}{a+b}$ times the ti triangle.

442. The sides of a triangle are 15, 41, and 52. Find the area of the two triangles formed by a bisecter of the angle between the first two sides.

METRIC

443. The sides of a triangle are 21, 25, and 30 centimeters. Find its area in square decimeters.

444. Find in meters, or decimals of a meter, the radii of the inscribed and circumscribed circles.

445. Find in millimeters the bisector of the angle between the first two sides.

446. Find in centimeters the median drawn to the second side.

447. Using the table of approximate metric equivalents (see Appendix), transform the results of the last four exercises into inches.

448. If the sides of a triangle are 9, 11, and 12 hectometers, find in hectares the area of the two triangles formed by a bisector of the angle between the first two sides.

449. A circular disc of wood is 6^{dm} in diameter and is covered with a veneer of wood, weighing 4 ounces to the square foot. What is the weight of the covering?

450. The wood veneer is 0.8 as heavy as water, and copper is 8.8 as heavy as water. What would be the weight of the above covering if made of copper?

451. A rectangular floor is laid in tiles in the shape of regular octagons, 16 octagons long and 12 octagons wide, each side of the octagons measuring 2 decimeters. Find in square meters the total area of the room, if the edges of the outer octagons just touch the edges of the floor.

452. A circular plot of ground is surrounded by a path whose area is equal to one-half the area of the plot. If the radius of the plot is 4.2 meters, find in decimeters the difference between the outer and inner edges of the path.

453. The legs of an isosceles triangle are each 8^{cm} in length, and the area equals $16^{\text{sq cm}}$. Find the number of degrees in each angle, and find in square millimeters the area of the inscribed circle.

EXERCISES

454. A cross-section of a train-house of a railway station is in the form of a pointed arch, 150 feet high, and composed of two circular arcs. The centres of the two arcs are on the ground, one on each side of a point directly beneath the apex of the arch. If the radius of each arc is 170 feet, find the width of the train-house.

455. How many kilometers is the light from a lighthouse 60 meters high visible?

456. Two wheels in the machinery of a factory are connected by a belt. The centres of the wheels are 90^{cm} apart, and the radii of the two wheels are 6^{dm} and 15^{cm}. How many revolutions of the small wheel will be made during one revolution of the large? How long a belt will it take to connect the wheels?

457. A safety bicycle with 28-inch wheels is geared by the use of a chain to 56. If there are 20 teeth upon the sprocket wheel, how many are there at the hub of the rear wheel? How many revolutions does each pedal make in traveling one mile? How many revolutions would be made in one mile by both pedals if the gear were 70? if the gear were 84? and if the number of teeth at the hub of the rear wheel were 8, how many would there be on the sprocket wheel?

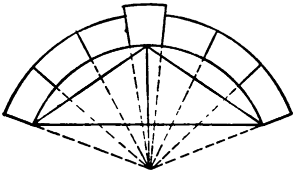
458. A bridge has a circular arch for its under surface. The span of the arch at the water level is 120 feet, and the top of the arch is $60(2-\sqrt{3})$ feet above the water. If the bridge is 30 feet wide, what is the area of that part of its under surface which is above water?

459. Find the radius of the earth if the top of a stake, which stands 8 inches out of the water, is just visible to a swimmer whose eye is at the surface of the water one mile distant.

460. The areas of two concentric circles are as $5:8$. The area of that part of the ring which is between two radii making an angle of $45°$ is 300 square feet. Find the radii of the two circles.

461. A quarter-mile running track has two parallel sides and semicircular ends. Each straight-away section is equal in length to one of the ends. If the track measures exactly one-fourth of a mile at the curb, or inner edge, how much distance does a runner lose in running two feet from the curb? six feet? What is the area of the track if it is 15 feet wide? What is the area of the enclosed field? What are the dimensions of a rectangular field sufficiently large to contain such a track?

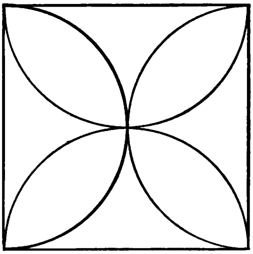
What will it cost at $2 per cubic yard to cover such a track with cinders to a depth of 2 inches?



462. Find the centre of an arch which is 6 feet high and 16 feet wide; *i.e.* which has a span of 16 feet.

NOTE. — The accompanying figure shows the method of cutting the stones in the formation of an arch.

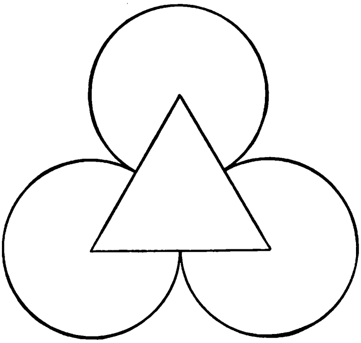
463. The inside dimensions of a circular, flat-bottomed dish are as follows: diameter of the bottom, 2.4^{dm}; diameter of the top, 336^{mm}; and the slant height of the sides, which are straight, is 9.6^{cm}. Within what distance of the bottom will a circular disc rest horizontally within the dish if the diameter of the disc is 312^{mm}?



NOTE.—The area of certain ornamental figures may be found by the addition or subtraction of certain parts of the figure, as

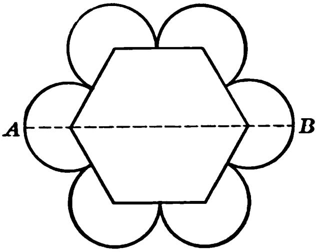
The sum of the four leaf-shaped figures in the figure equals the area of the four semi-circles minus the area of the square.

464. Find the area of four such lobes in a square whose side equals 14.

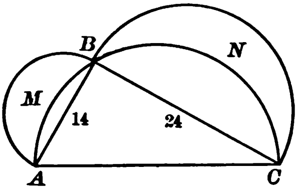


465. Find the area of a *tre-foil*, the construction of which appears in the figure, if the distance between the centres equals 5 centimetres.

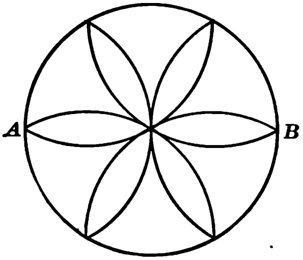
466. The extreme dimension (*AB*) of a six-lobe rose-window is 28 decimeters. Find its area.



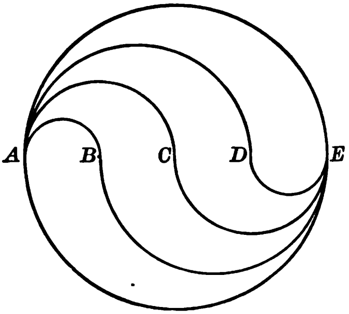
467. Find the area of the two crescents formed by the semi-circumferences described upon *AC*, *AB*, and *BC* as diameters, and compare the sum of their areas with the area of the right triangle *ABC*.

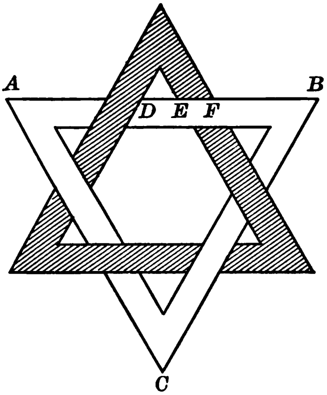


468. The diameter of a circle *AB* is 8. The circumference is divided into six equal parts and lobes formed as indicated. Find the area of the six lobes.



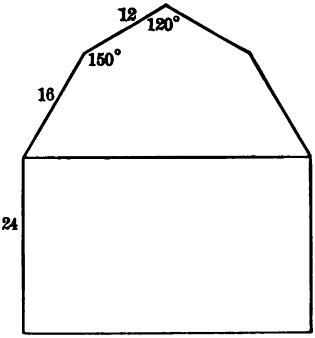
469. *AE*, a diameter, is divided into any number of equal parts and semi-circumferences drawn as indicated. Find in terms of π the areas of the resulting figures.



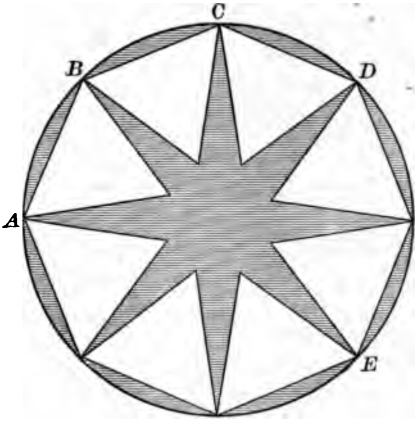


470. $AB = 18$, $DE = 2$.
$FB = 6$.

Find the area of the shaded portion.



471. A house is 42 feet long and the ends of the shape and dimension indicated in the figure. Find its entire exterior surface. Find also its total height.

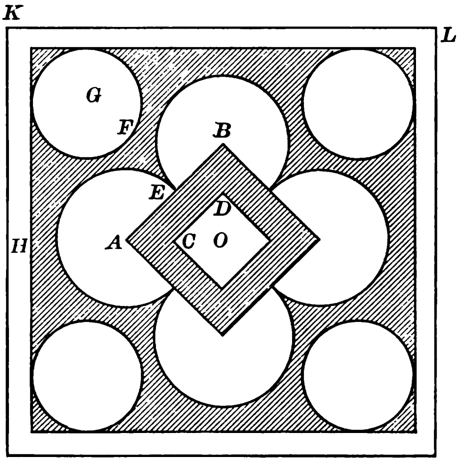


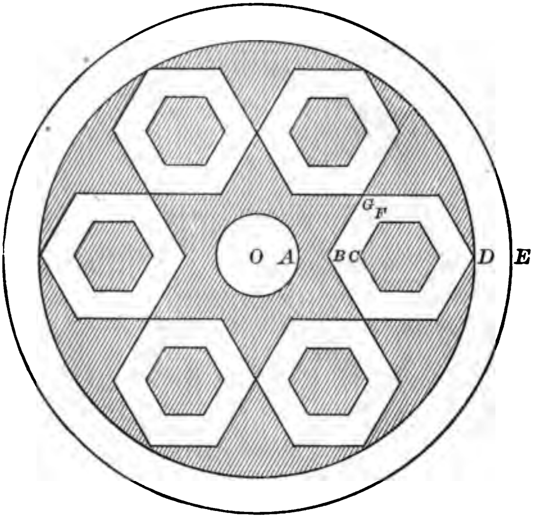
472. The perimeter of an inscribed regular octagon ADE is $16\sqrt{3}$. On each side are drawn equilateral triangles. Find the area of the shaded portions of the figure.

473. $KO = 12\sqrt{2}$
$AO = 6$
$CD = \frac{1}{2} AB$
$GF = \frac{5}{2}\sqrt{2}$

The width of the border $H = 2$

Find the area of the shaded portions.





474. $OA = AB = BC = DE = 3$. Find the area of the portions which are not shaded.

EXERCISES INVOLVING LOGARITHMS

475. If the diameter of a circle is 36.43^{cm}, what is the length of an arc of $77° 33'$?

476. If the radius of a circle is $17\sqrt{2.461}$, what is the area of a sector whose angle is $148° 10' 42''$?

477. If the radius of a circle is 338.92^{mm}, what is the area of a segment whose arc is 120°?

478. The side of an equilateral triangle is 12.374^{cm}. Find in square millimeters the area of the inscribed and circumscribed circles.

479. The perimeter of a regular hexagon circumscribed about a circle is $1450\sqrt{3}$. What is the circumference of the circle?

480. A plot of ground is in the shape of a regular decagon, and the length of each side 176.4^{m}. Find its area in hectares.

481. The diameter of a carriage-wheel is 1.1875^{m}. How many revolutions does it make in traveling $\frac{1}{4}$ of a mile?

482. What is the width of the ring between two concentric circumferences whose lengths are 456.36 feet and 408.67 feet?

483. The radius of a circle is 6.3472^{dm}. Find the angle and area of a sector whose arc is 4.7335^{dm}.

484. If a meridian of the earth (considered a sphere) is $40{,}000{,}000^{\text{m}}$, find the length of an arc of $1''$.

485. If a meridian of the earth is $40{,}000{,}000^{\text{m}}$, find the radius in kilometers.

486. A circular lot of land contains 2.2794^{Ha}. What must be the width of an encircling walk containing $\frac{1}{4}$ of a hectare?

487. The radius of a circle is 3.444 feet. Find the circumference of a circle 8.165625 times as large.

488. The radius of a circle is 3.375^{cm}, and two tangents are drawn from an outside point making an angle of 60°. Find in square millimeters the area enclosed by the tangents and the arc.

489. The diameter of a circle is $416.6\frac{2}{3}$. Two sides of an inscribed triangle are 250 and 400. Find its third side; its area.

490. In the same triangle find the bisector of the angle between the given sides; the median to the third side; the radius of an inscribed circle.

491. Find the area of the two triangles formed by a bisector of the angle between the first two sides.

492. If the radius of the earth is 3963 miles, find the length of a parallel of latitude 60° from the equator.

493. A boy, whose eye is 4 feet 6 inches from the ground, can just see the top of a steeple when he stands back $3\frac{1}{2}$ feet from a fence 6 feet high. The distance from the foot of the fence to the centre of the base of the steeple is $210\frac{3}{4}$ feet. Find the height of the steeple.

494. The radius of a circle is 3.225. Find the area of the regular inscribed decagon.

495. The radius of a circle is 1.212. Find the side of the regular inscribed pentagon; the diagonal.

496. The length of the arc subtended by the side of a regular inscribed dodecagon is 4.2648. Find the area of the circumscribed equilateral triangle.

497. Quebec is $5^\circ 2'$ due north of Providence, R. I. How many miles apart are they if the radius of the earth is 3963 miles?

498. A circle and a square have each a perimeter of 242.15 feet. Which has the greater area, and how much?

BOOK V

CONSTRUCTION OF GEOMETRICAL FIGURES

INTERSECTION OF LOCI

319. Upon the basis of the principles established in the preceding books, it is possible to make geometric construction of the figures used to illustrate and prove the various theorems. In *geometric* construction, the principle of the radius and of the straight line only may be used, and no appliances except dividers and a straight-edge may be employed. In *mechanical* construction, the linear scale, measured upon a straight-edge, an angle scale, measured upon a protractor, triangles such as an isosceles right triangle or a 30° and 60° right triangle, a parallel ruler, etc., may profitably be employed.

In simple problems the data furnished in the given statement are usually sufficient to indicate at once the process of construction. In more difficult problems a process of analysis may be necessary. Assume the figure to be completely drawn and appreciate the relations between the parts of the figure; on the basis of these relations proceed with the construction.

In the geometric construction of figures, the position of a point is often determined by the intersection of two lines, and frequently these lines are *loci*, according to the conditions specified.

Nothing is of more value in the construction of geometric figures than a familiarity with the common loci and a habit of determining points by the intersection of these loci.

Loci

320. The following loci have found illustration and proof among the theorems and corollaries of the preceding books:

1. The locus of a point equidistant from two given points is the perpendicular bisector of the line joining the two given points.

2. The locus of a point equidistant from two lines which are not parallel is the bisector of the angle between the lines.

3. The locus of a point at a given distance from a given straight line is a pair of parallels to the given line, one on either side of the line, and at the given distance from it.

4. The locus of a point at a given distance from a given point is the circumference described about the given point as a centre, with the given distance as a radius.

321. The following loci should be determined and the authority or reason for the locus established; note also the impossible and the limiting cases:

5. Locus of a point equidistant from two given parallels.

6. Locus of a point equidistant from two given intersecting straight lines.

7. Locus of the (opposite) vertices of all triangles that have a common base and equal altitudes.

8. Locus of the vertices of all triangles that have the same base and a given vertical angle.

9. Locus of the vertices of all isosceles triangles which have the same base.

10. Locus of the vertices of all right triangles that have the same hypotenuse.

11. Locus of a point at a given distance from a given circumference; at a radius distance.

12. Locus of the centres of circumferences which pass through two given points.

13. Locus of the centres of circles of given radius which touch a given line.

14. Locus of the centres of circles which touch a given line at a given point.

15. Locus of the centres of circumferences which have a given radius and pass through a given point.

16. Locus of the centres of circles which touch two given parallels.

17. Locus of the centres of circles which touch two given intersecting lines.

18. Locus of the centres of circles which have a given radius and touch a given circle.

19. Locus of the centres of circles which touch a given circle at a given point.

20. Locus of the centres of circles which touch two concentric circumferences.

21. Locus of the middle points of parallel chords in a given circle.

22. Locus of the middle points of all chords in a given circle which have a given length.

23. Locus of the middle points of all chords in a given circle which pass through a given point.

24. Locus of the middle points of all chords which are drawn from a given point on the circumference.

25. Locus of the middle points of all secants drawn from a given point to a given circumference.

26. Locus of the middle point of a moving straight line whose extremities touch the sides of a right angle.

27. Locus of the end points of tangents of a given length to a given circle.

28. Locus of the end points of a series of equal parallels drawn to a straight line on the same side of it.

29. Locus of the end points of a series of equal parallels, perpendicular to the diameter and which meet without intersecting a given semi-circumference.

30. Locus of points at which tangents to a given circle form equal angles.

31. An angle *ABC* is inscribed in a given segment of a circle. *AB* is produced distance equal to itself to a point *D*. What is the locus of *D*?

In the following, the loci should be established but the proofs may be omitted:

32. Locus of the centre of a circle inscribed in a right triangle of given hypotenuse.

33. A triangle has a given base and a given vertical angle. Find the centre of the inscribed circle.

34. Find the locus of a point the sum of whose distances from two given lines equals a given constant.

35. Find the locus of a point the difference of whose distances from two given lines equals a given constant.

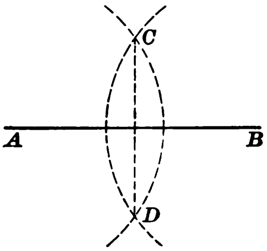
36. Locus of points the distances of which from two given parallels are as $m : n$; for example, as $2 : 1$.

37. Locus of points the distances of which from two intersecting lines are as $m : n$; for example, as $2 : 1$.

PROBLEMS OF CONSTRUCTION

PROPOSITION I

322. Problem. *To bisect a given straight line.*



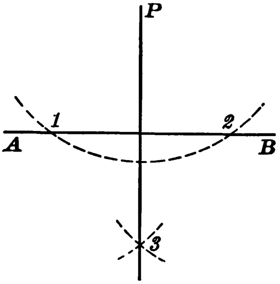
Appl. To bisect *AB*

Cons. From *A* and *B* as centres, with radius greater than the apparent length of $\frac{1}{2}$ *AB,* describe intersecting ⓢ

Dem. *CD* bisects *AB* [two pts. *C* and *D*, = dist. from *A* and *B*, determine ⊥ bis. of *AB*]

PROPOSITION II

323. Problem. *From a given point without a straight line to draw a perpendicular to that line.*



NOTE. — Observe the figure carefully, discover the process, and give the necessary proof. The process of construction must begin at the point *P*, as the only given fixed point.

PROPOSITION III

324. Problem. *From a given point upon a straight line to erect a perpendicular to that line.*

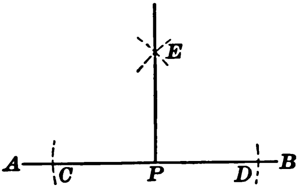


FIG. 1.

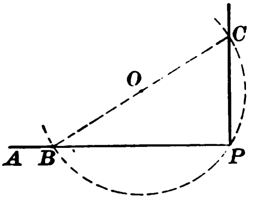


FIG. 2.

Appl. To erect $\perp$ to AB at P

Cons. With P as centre (Fig. 1) describe ⌒s cutting AB at C and D. With C and D as centres, with radius $> CP$ describe ⌒s intersecting at E. Draw EP

Dem. $EP \perp AB$ [E and P = dist. from C and D]

Second Method

Cons. With any convenient pt. O, not in AP, (Fig. 2) as a centre, and a rad. $= OP$, describe an arc cutting AP at B. Draw diameter BOC. Draw CP

Dem. $CP \perp AP$ [CPB = rt. $\angle$ (insc. in semi ⊙)]

Sch. This method is of value when the point is at or near the end of the given line.

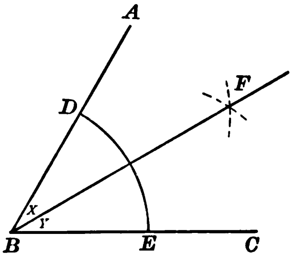
EXERCISES

499. In a given line find a point which is equidistant from two given points. The $\perp$ bis. of a line joining the two given points is the locus of all points which are equidistant from the given points.]

500. Construct a circumference of a given radius which shall pass through two given points. [The $\perp$ bis. of the line joining the points is the locus of what ?]

Proposition IV

325. Problem. *To bisect a given arc, or a given angle.*



Appl. Let $B =$ vertex of $\angle ABC$ and also the centre of an arc DE

Cons. To bisect the arc draw its chord and construct its $\perp$ bis. The $\perp$ bis. also bisects the arc.

To bisect the $\angle$, with B as a centre describe any arc as DE, then bisect the arc

Dem. $X = Y$ [intercept = $\overset{\frown}{s}$]

EXERCISES

501. Given an angle, to construct its complement.

502. Given an angle, to construct its supplement.

503. To construct an angle of 45°; of 135°; of 22½°; of 67½°.

504. To construct an angle of 60°; of 120°; of 30°; of 150°.

505. Construct a rectangle whose area = 1⅞ sq. inches and such that its base is ⅝ of its altitude.

506. Given the diagonal of a square, to construct the square.

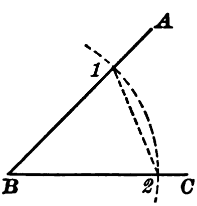
507. Given the base of an isosceles triangle equal to 4½ centimetres and its altitude equal to 40 millimetres, to construct the triangle.

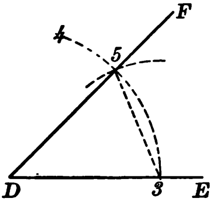
508. To circumscribe a circle about a given rectangle.

509. To construct a chord in a given circle, having given the middle point of the chord.

PROPOSITION V

326. Problem. *At a given point in a straight line, to make an angle equal to a given angle.*





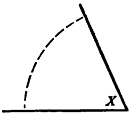
Appl. At D in DE to make an ∠ = B

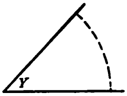
Cons. With B as centre describe ⌒ *1 2*
“ D “ “ “ ⌒ *4 3*, having same radius $B1$
With *3* as centre and radius *3 5*, = *2 1*, describe ⌒ intersecting *3 4* at *5*
Draw DF through *5*

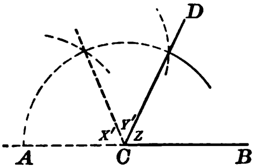
Dem. The ⌒s are = (= chds. in = ⓢ)
∴ “ ∠s “ =

PROPOSITION VI

327. Problem. *Given two angles of a triangle, to find the third.*







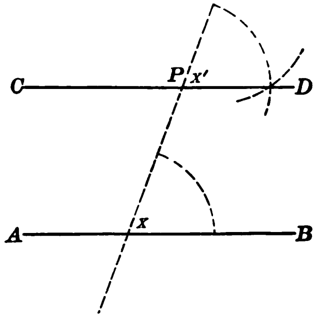
Appl. If X and Y are two ∠s of a △, to find the third

Cons. At C, any pt. in a line AB, make $X' = X$, and $Y' = Y$

Dem. Z = third ∠ of △ [?]

PROPOSITION VII

328. Problem. *Through a given point to draw a parallel to a given straight line.*



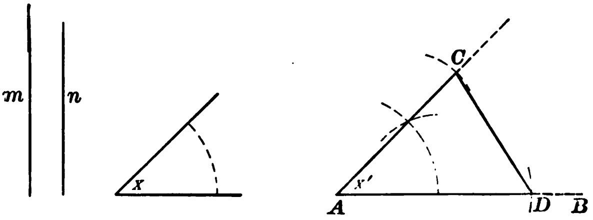
Appl. At P to draw a ∥ to AB

Cons. Draw PD, making $x' = x$

Dem. $CD \parallel AB$ [corresp. ∠s =]

PROPOSITION VIII

329. Problem. *Given two sides and the included angle of a triangle, to construct the triangle.*



Appl. To construct △, having given two sides, m and n, and x, their included ∠

Dem. Discover and prove the process

Proposition IX

330. Problem. *Given a side and two angles of a triangle, to construct the triangle.*

From an indefinite line lay off a portion = to the given side. At each end of this side construct $\measuredangle$s = respectively to the given $\measuredangle$s.

Proposition X

331. Problem. *Given three sides of a triangle, to construct the triangle.*

Make a line equal to one side, and from either end with radii equal respectively to the other two sides describe intersecting arcs.

Proposition XI

332. Problem. *Given two sides of a triangle and the angle opposite one of them, to construct the triangle.*

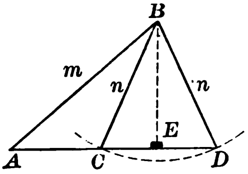


Fig. 1

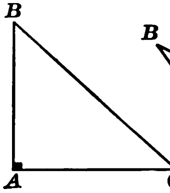


Fig. 2

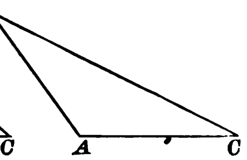


Fig. 3

Appl. To construct a $\triangle$, having given $\angle A$, side m, and a side n opposite $\angle A$

Cons. At A, in indef. line AD, make $\angle$ = to given $\angle$. Lay off $AB = m$. With B as a centre and radius $= n$ draw $\frown$ cutting AD at C and D. ABC, ABD, etc., will be the required $\triangle$, according to size of the $\angle A$ and the relative lengths of m and n.

Dem. 1. When A is acute.

If $n < m$ but $> \perp BE$, both $\triangle$ ABC and ABD satisfy the conditions.

If $n = \perp BE$, the rt. $\triangle$ ABE is the required $\triangle$.

If $n < \perp BE$, no solution is possible.

If $n > m$, the intersecting $\frown$ will cut the base but once on the right of A, and one $\triangle$ is possible.

2. If the $\angle A$ is a rt. $\angle$ (Fig. 2) or an obtuse $\angle$ (Fig. 3), n must be $> m$, and one solution only is possible.

Proposition XII

333. Problem. *To inscribe a circle in a given triangle.*

The centre of the inscribed $\odot$ is the intersection of the bisectors of any two angles (118).

Proposition XIII

334. Problem. *To circumscribe a circle about a given triangle.*

The centre of the circumscribed $\odot$ is the intersection of the $\perp$ bisectors of any two sides (119).

335. Sch. I. To pass a $\odot$ through any three given points not in the same straight line, connect the points, and proceed as in the above construction.

336. Sch. II. The centre of any arc may be found by erecting $\perp$ bisectors of any two chords of the arc.

EXERCISES

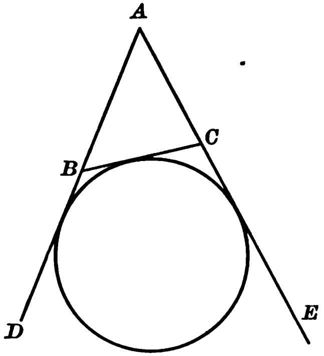
510. To trisect a given right angle.

511. To construct a parallelogram, having given the two diagonals and their included angle.

512. To construct a parallelogram, having given two sides and the included angle.

PROPOSITION XIV

337. Problem. *To escribe* a circle upon a side of a given triangle.*



Appl. To escribe a ⊙ upon the side *BC* of the △ *ABC*

Cons. Produce *AB* and *AC*.

Dem. The centre of the escribed ⊙ is the intersection of the bisectors of the exterior ∠s *DBC* and *ECB* of the △

PROPOSITION XV

338. Problem. *To draw a tangent to a circle through a given point on the circumference.*

Erect a perpendicular from the extremity of a radius drawn to the given point.

Ex. 513. To construct a parallelogram, having given a side and the diagonals.

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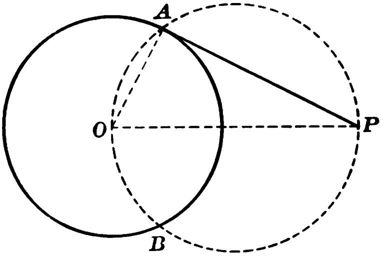
* An *escribed* circle is tangent to one side of a triangle externally, and to the other two sides produced.

PROPOSITION XVI

339. Problem. *To draw a tangent to a circle from a point outside the circumference.*



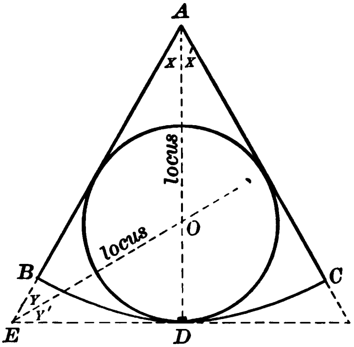
Cons. Connect the point P with centre of ⊙.
Take OP as diam. of a ⊙ intersecting given ⊙ at A and B

Dem. PA or PB = tangent

[Draw OA. OAP = rt. ∠ (insc. in semi ⊙. then AP is ⊥ to rad.)]

PROPOSITION XVII

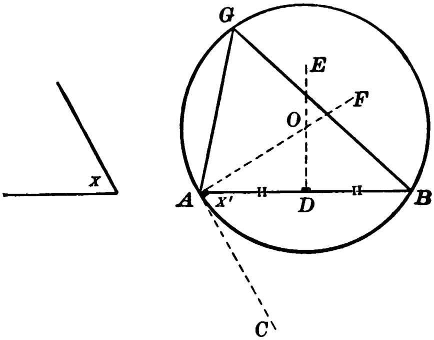
340. Problem. *To inscribe a circle in a given sector.*



Cons. Bisect ∠ A.
Bisect ∠ E, between a radius produced and a tangent to the arc of the sector at D

PROPOSITION XVIII

341. Problem. *Upon a given straight line to construct a segment in which an inscribed angle shall be equal to a given angle.*



Appl. On AB to construct a segment in which an inscribed $\angle$ shall $= X$

Cons. At A make $X' = X$ and draw $AF \perp AC$.
Erect DE, $\perp$ bis. of AB

Dem. $O =$ centre and $OA =$ rad. of required segment
$G = X'$ [each meas. by $\frac{1}{2} AB$]
$G = X$

Sch. The arc AGB is the locus of the vertices of all triangles having the base AB and a vertical $\angle = X$

EXERCISES

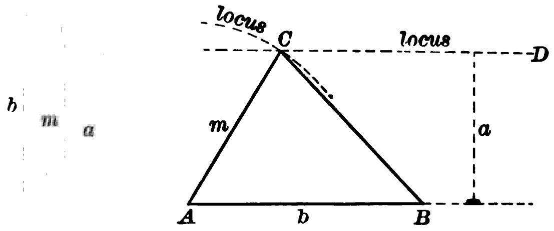
514. To draw a line parallel to a given straight line and tangent to a given circle.

515. To draw a line perpendicular to a given straight line and tangent to a given circle.

516. Through a given point within a circle to draw a chord equal to a given chord.

PROPOSITION XIX

342. Problem. ***To construct a triangle, having given the base, an adjacent side, and the altitude of the triangle.***



Appl. $b =$ given base

$a =$ " alt.

$m =$ " side

Cons. Make $AB = b$.

Draw $CD \parallel$ to AB at dist. a

[$CD =$ locus of the vertices of all $\triangle$ having base AB and alt. a]

With A as centre and rad. m, describe $\frown$ intersecting CD at C.

[Arc intersecting CD at C is the locus of the vertices of all $\triangle$ having base AB and adjacent side m]

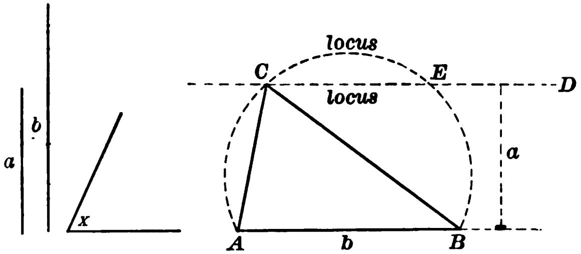
ABC is the triangle.

The point C, the intersection of two loci, is the vertex of the required triangle.

NOTE. — In this and the few following problems is illustrated the process of determining a required point by the intersection of certain *loci* dependent upon the conditions of the problem.

PROPOSITION XX

343. Problem. *To construct a triangle having given the base, the altitude, and the vertical angle.*

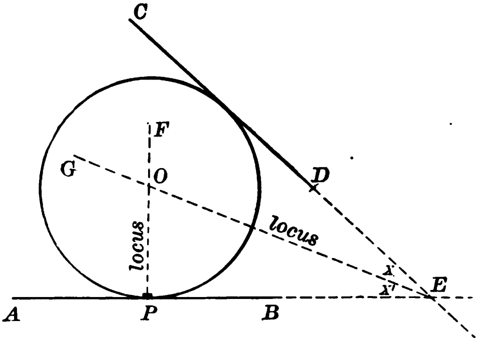


Cons. Construct upon given base b a segment in which an inscribed ∠ shall $= x$ (341). Draw $CD \parallel AB$ at dist. a

Dem. The arc of the segment and the parallel CD are the loci.

PROPOSITION XXI

344. Problem. *To describe a circle touching two given straight lines, one of them at a given point.*

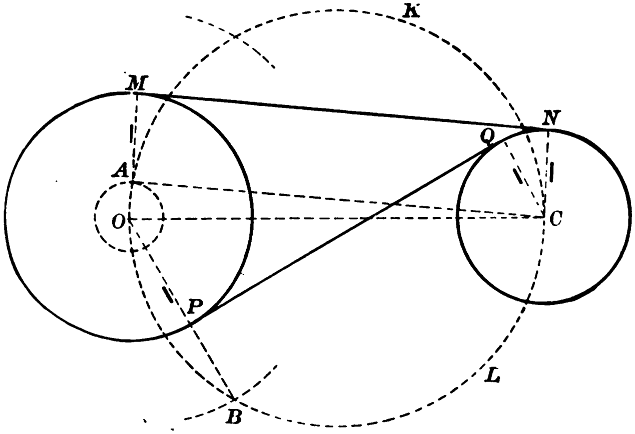


Appl. To draw ⊙ touching CD at any point, and AB at P

Dem. O is the required centre, and OP the radius
What locus is GE and what locus is FP?

PROPOSITION XXII

345. Problem. *To draw a common tangent to two given circles.*



Appl. To draw MN

Cons. Upon OC as diam. describe OKL.

About O as a centre with rad. $OA, =$ dif. of given radii, describe $\odot$ intersecting OKL at A.

Draw OA and produce to M. Draw $CN \parallel OM$

Dem. [Hints to proof]

$OAC =$ rt. $\angle$ [insc. in semi $\odot$]

$MNCA =$ rectangle [$MA =$ and $\parallel$ to NC, etc.]

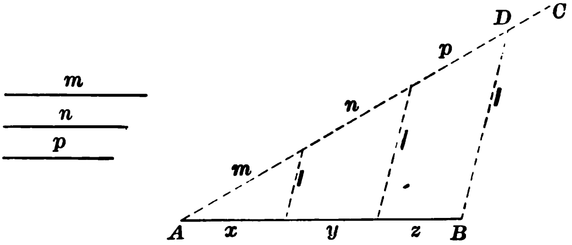
$MN \perp$ at ext. of radii

Cons. To draw PQ

Describe $\frown$ with O as centre, and radius $=$ sum of given radii intersecting OKL at B. Draw OB, and $CQ \parallel PB$

PROPOSITION XXIII

346. Problem. *To divide a given straight line into parts proportional to any number of given lines.*



Appl. To divide AB into parts proportional to m, n, and p

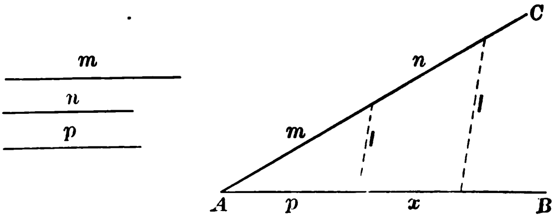
Cons. Draw any line AC. Measure $m = m$, $n = n$, etc.
Join DB, and through pts. of division draw ∥s to DB

Dem. x, y, and z are the required parts of AB

Cor. By measuring any number of = distances upon AC, AB may be divided into that number of *equal* parts.

PROPOSITION XXIV

347. Problem. *To construct a fourth proportional to three given lines.*

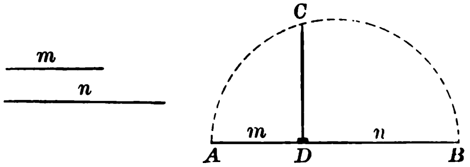


Cons. Upon indef. lines AB and AC, lay off $m = m$, $n = n$, and $p = p$. Draw ∥s as indicated, obtaining x

Cor. By making n and $p =$, a third proportional to m and n is found.

PROPOSITION XXV

348. Problem. *To construct a mean proportional between two given lines.*



Appl. To construct a mean proportional between m and n

Cons. Take $AB = m + n$.

Describe semi ⊙ upon AB.

Erect $DC \perp AB$ at point D

Dem. $CD =$ mean prop'l between m and n [$ACB =$ rt. $\triangle$, and $\perp$ to hyp. from vert. of rt. $\angle =$ mean prop'l bet. seg. of hypot.]

349. Sch. A mean proportional between a line 5 units long and another 1 unit long is a line whose length $= \sqrt{5}$, etc.

EXERCISES

517. Inscribe in a given circle a triangle similar to a given triangle.

518. Circumscribe about a given circle a triangle similar to a given triangle.

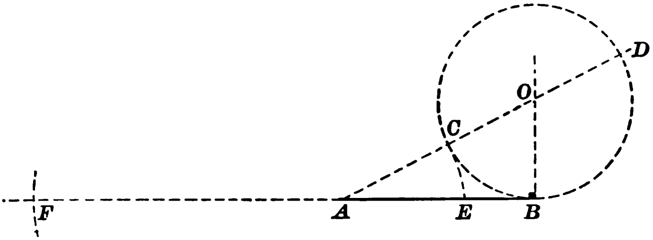
519. Construct a line $= \sqrt{12}$. [Make three different applications of the same process, using the same units of measurement, and compare the three resulting lines.]

520. Construct a triangle of given perimeter similar to a given triangle.

521. Construct a right triangle, having given its perimeter and an acute angle.

PROPOSITION XXVI

350. Problem. *To divide a given line in extreme and mean ratio.*



Appl. To divide AB (internally) so that $AB : AE = AE : EB$,
and (externally) " " $AB : AF = AF : FB$.

Cons. At B erect $\perp BO = \frac{1}{2} AB$.
Describe $\odot\, CBD$.
Draw secant AD through O.
Measure $AE = AC$ and $AF = AD$

Dem. 1. $AD : AB = AB : AC$ [tang. = m. p. bet. secant and its external seg.]

$AD - AB : AB = AB - AC : AC$ [division]

$AE : AB = EB : AE$ [subs. = for = (CD for AB, etc.)]

$AB : AE = AE : EB$ [inversion]

2. $AD : AB = AB : AC$ [as above]

$AD + AB : AD = AB + AC : AB$ [composition]

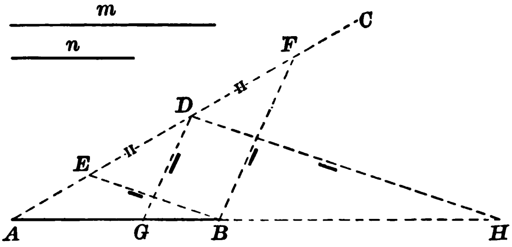
$FB : AF = AF : AB$ [subs. = for =]

or $AB : AF = AF : FB$ [invert and exchange couplets]

Sch. See section 284.

PROPOSITION XXVII

351. Problem. *To divide a given line harmonically in a given ratio.*



Appl. To divide AB, both internally and externally, into segments proportional to m and n

Cons. Let AC make any $\angle$ with AB
Measure $AD = m$, and then
DF and $DE = n$
Draw FB and EB
Draw $DG \parallel FB$
$DH \parallel EB$
Then $GA : GB = HA : HB$

Dem. $\frac{GA}{GB} = \frac{DA}{DF} = \frac{m}{n}$ [∥ to side of $\triangle AFB$]

$\frac{HA}{HB} = \frac{DA}{DE} = \frac{m}{n}$ [∥ to side of $\triangle AHD$]

$\frac{GA}{GB} = \frac{HA}{HB}$ [Ax. 1]

Sch. See section 203.

Ex. 522. To divide the side of a triangle harmonically in a ratio of 3 to 2.

EXERCISES IN CONSTRUCTION

To construct a triangle, having given

523. The base, altitude, and an adjacent angle.

524. The base, altitude, and the vertical angle.

525. The base, vertical angle, and the median to the base.

526. A side, an adjacent angle, and the sum of the other sides.

527. A side, an adjacent angle, and the difference of the other sides.

528. The perimeter and the angles.

529. The angles and the radius of the inscribed circle.

530. The angles and the radius of the circumscribed circle.

531. A side, an adjacent angle, and the radius of the circumscribed circle.

To construct a right triangle, having given

532. The hypotenuse and a leg.

533. The hypotenuse and the altitude upon the hypotenuse.

534. The median to the hypotenuse and the altitude upon the hypotenuse.

535. An acute angle and the radius of the inscribed circle.

536. A leg and the radius of the inscribed circle.

537. An acute angle and the difference of the legs.

538. An acute angle and the sum of the hypotenuse and the leg opposite the given angle.

To construct an isosceles triangle, having given

539. The base and the vertical angle.

540. A leg and a base angle.

541. The altitude and the vertical angle.

542. The base and the radius of the inscribed circle.

To construct an equilateral triangle, having given

543. A side.

544. The perimeter.

545. The altitude.

546. The radius of the inscribed circle.

547. The radius of the circumscribed circle.

To construct a trapezoid, having given

548. The four sides.

549. The bases and the diagonals.

To construct an isosceles trapezoid, having given

550. The bases and a diagonal.

551. The bases and the altitude.

552. The bases and the radius of the circumscribed circle.

To construct a rectangle, having given

553. A side and an angle between the diagonals.

554. The perimeter and the ratio between the sides.

To construct a rhombus, having given

555. The diagonals.

556. A side and the radius of the inscribed circle.

557. An angle and a diagonal.

To construct a circumference

558. Which has a given radius and which touches two intersecting lines.

559. Which has a given radius and which touches two given circles.

560. Which has a given radius and touches a given straight line and also a given circumference.

561. Which touches a given circumference at P, and passes through a given point Q.

562. Which touches two parallel straight lines and which passes through a given point.

563. Which touches two given straight lines, and passes through two given points.

564. Which is tangent to three given lines.

565. Which passes through two points and has its centre in a given line.

566. Which is tangent to two intersecting circles.

567. To construct three equal circles within a given circle, each touching the other two and also the given circle.

568. Which touches two concentric circumferences and passes through a given point.

PROPOSITION XXVIII

352. Problem. *To construct a square equivalent to a given rectangle or parallelogram.*

A mean proportional between the altitude and the base of the given rectangle or parallelogram will be the side of the required square.

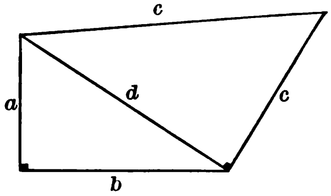
PROPOSITION XXIX

353. Problem. *To construct a square equivalent to a given triangle.*

A mean proportional between the base and one-half the altitude will be the side of the required square.

PROPOSITION XXX

354. Problem. *To construct a square equivalent to the sum of two or more given squares.*



Construct a right triangle with legs equal to a and b, the sides of the given squares. d will be side of the square equivalent to their sum. The process may be continued as indicated in the figure.

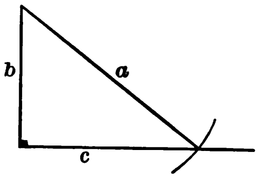
EXERCISES

569. The sides of a triangle are 15, 17, and 20. Construct an equivalent square.

570. Construct a square equivalent to the sum of three squares, whose sides are respectively 5, 8, and 9.

PROPOSITION XXXI

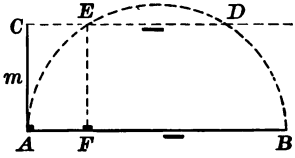
355. Problem. *To construct a square equivalent to the difference of two given squares.*



Let a and b be the sides of the two given squares. Construct the right triangle as indicated. c will be the side of the required square.

PROPOSITION XXXII

356. Problem. *To construct a rectangle or parallelogram which shall be equivalent to a given square, and have the sum of its base and altitude equal to a given line.*



Let AB be the line = sum of base and altitude.

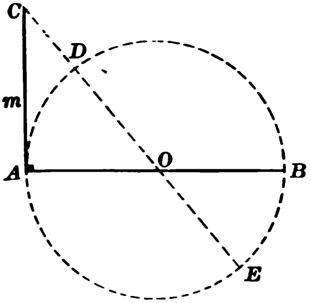
Erect $\perp$ $AC = m$, side of given square. Draw semi ⊙.

Draw $CD \parallel AB$ and $EF \perp AB$

AF and FB are the two dimensions of the rectangle or parallelogram $\left[\begin{array}{l} EF (= m) = \text{mean prop'l} \\ \text{between } AF \text{ and } FB \end{array}\right]$

PROPOSITION XXXIII

357. Problem. *To construct a rectangle or a parallelogram which shall be equivalent to a given square and have the difference between its base and altitude equal to a given line.*



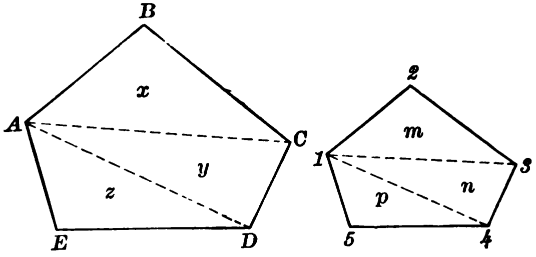
Let AB = dif. of base and altitude, m = side of given square.

Draw lines as indicated.

CD and CE = two dimensions of rectangle or parallelogram [tang. AC = mean prop'l between whole secant and external segment.]

PROPOSITION XXXIV

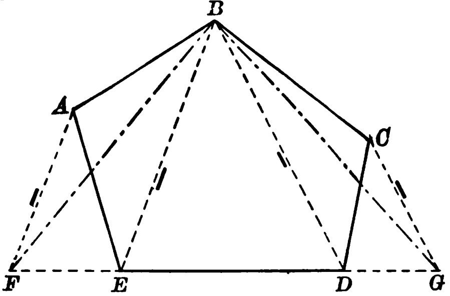
358. Problem. *Upon a given line to construct a polygon similar to a given polygon.*



See section 210.

PROPOSITION XXXV

359. Problem. *To construct a triangle equivalent to a given polygon.*



To construct a △ ≡ given polygon $A - E$

Draw diagonal BE

Draw $AF \parallel BE$, meeting DE produced

△ BFE ≡ △ BAE [same base BE and same altitude (⊥ distance between ǁs)]

⬠ $BCDF$ ≡ ⬠ $A - E$, and the number of sides is reduced by one. [subs. BFE for BAE]

Continue the process as long as necessary.

△ FBG ≡ given ⬠ of five sides.

360. Sch. A square may be constructed equivalent to a given triangle (353); therefore a square may be constructed equivalent to a given polygon and, by the use of the right triangle (354, 355), a square may be constructed equivalent to the sum or difference of any number of given polygons.

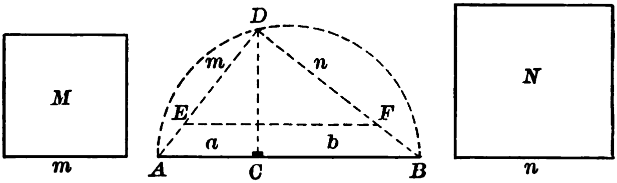
EXERCISES

571. Construct a square equivalent to the sum of a given triangle and a given hexagon.

572. Construct a square equivalent to the difference of a given trapezium and a given pentagon.

PROPOSITION XXXVI

361. Problem. *To construct a square having a given ratio to a given square.*



To construct a square having to the given square a given ratio $b : a$

Cons. Divide a line AB at C into parts which have the given ratio. Describe semi ⊙ upon AB as diam.

Erect ⊥ DC. Connect DA and DB.

Measure $DE = m$, side of given square

Draw $EF \parallel AB$. $DF = n$, the side of the required square

Dem. $\frac{\overline{DA}^2}{\overline{DB}^2} = \frac{a}{b}$ [sq. of legs of rt. △ are as seg. of hyp. formed by ⊥ from vertex of rt ∠ (216)]

$\frac{m^2}{n^2} = \frac{a}{b}$ $\left[\frac{m}{n} = \frac{DA}{DB}\text{ and squaring}\right]$

$\therefore$ n = side of square having ratio $b : a$ to given square

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EXERCISES

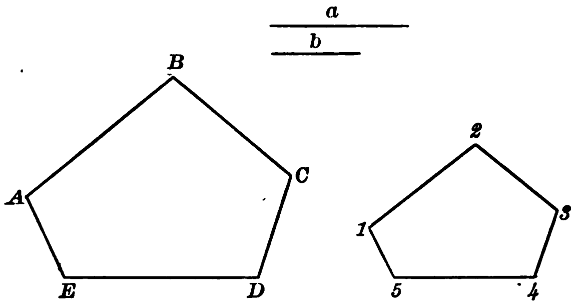
573. To draw a parallel DE to a side BC of a triangle ABC, such that $DE = EC$.

574. To draw a parallel DE to a side BC of a triangle ABC, such that $DE = BD + EC$.

575. Through a given point without a given circle, to draw a secant whose internal and external segments shall be equal.

PROPOSITION XXXVII

362. Problem. *To construct a polygon similar to a given polygon and having a given ratio to it.*



To construct a ⬠ sim. to $A-E$ and having ratio to it of b to a

Cons. Find 54 as in the process of the previous problem, such that $\overline{54}^2 : \overline{ED}^2 = b : a$

Upon 54 construct ⬠ sim. to $A-E$

Dem. $\frac{1-5}{A-E} = \frac{\overline{54}^2}{\overline{ED}^2} = \frac{b}{a}$ [why ?]

PROPOSITION XXXVIII

363. Problem. *Two similar polygons being given, to construct a similar polygon equivalent to their sum, or to their difference.*

Find by use of the right triangle (354, 355) a line whose square ≡ sum or the difference of two homologous sides of the given polygons.

Upon this line construct a polygon similar to the given polygons.

Proposition XXXIX

364. Problem. *To inscribe a square in a circle.*

Draw two diameters $\perp$ to each other.

Join the points upon the circum.

The figure is a square [four pts. of = division]

365. Cor. Bisect the $\overset{\frown}{s}$ subtended by the sides of the square to form an octagon, and so on for polygons of 16, 32 sides, etc.

Proposition XL

366. Problem. *To inscribe a regular hexagon in a circle.*

Lay off chords = radius of the circle.

The figure is a regular hexagon according to 281.

367. Cor. Bisect the $\overset{\frown}{s}$ subtended by the sides of the hexagon for polygons of 12 sides, etc.

Proposition XLI

368. Problem. *To inscribe a regular decagon in a given circle.*

Divide the radius into extreme and mean ratio; the longer segment = side of reg. decagon, as proved in 284.

369. Cor. Bisect $\overset{\frown}{s}$, etc., for polygons of 20 sides, etc.

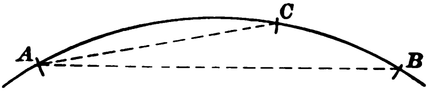
EXERCISES

576. To inscribe in a semicircle a rectangle similar to a given rectangle.

577. In an equilateral triangle to construct three equal circles, tangent to each other and each circle tangent to two sides of the triangle.

Proposition XLII

370. Problem. *To inscribe a regular pentedecagon in a given circle.*



Find $\frown AB = \frac{1}{6}$ circumference, and
" $\frown AC = \frac{1}{10}$ "
Then $CB = \frac{1}{6} - \frac{1}{10} = \frac{1}{15}$ circumference,
and chd. $CB =$ side of pentedecagon.

Proposition XLIII

371. Problem. *To circumscribe about a circle a polygon similar to a given regular inscribed polygon.*

Draw tangents to ⊙ at vertices of given ⬠;
or, draw tangents at middle pts. of ⌒s subtended by sides of given ⬠.

EXERCISES

578. To bisect a given circle by a concentric circumference.

579. To divide a given circle by concentric circumferences into any odd number of equivalent parts, as three, or five, etc.

580. To construct a right triangle equivalent to a given triangle.

581. To construct upon a given base an isosceles triangle equivalent to a given triangle.

582. To bisect a triangle by a line parallel to the base.

583. To bisect a triangle by a line perpendicular to the base.

584. To bisect the area of a parallelogram by a line passing through a given point.

585. To bisect the area of a trapezoid by a line perpendicular to the bases.

586. To bisect the area of a trapezoid by a line passing through a given point in the longer base.

587. To construct a square equivalent to $\frac{3}{4}$ of a given square.

588. To construct a regular hexagon equivalent to $\frac{2}{3}$ of a given regular hexagon.

589. To inscribe in a given circle five equal circles, such that each circle is tangent to two other circles and also internally to the given circle.

590. To construct a circumference equal to the sum of two given circumferences.

591. To construct a circle equivalent to $\frac{4}{9}$ of the sum of two given circles.

COLLEGE EXAMINATIONS.

HARVARD COLLEGE, *June*, 1892.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that if two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the lesser side.

In a certain right triangle one of the legs is half as long as the hypotenuse; what are the angles of the triangle?

2. Show how to find on a given indefinitely extended straight line in a plane, a point O which shall be equidistant from two given points A, B in the plane. If A and B lie on a straight line which cuts the given line at an angle of 45° at a point 7 in. distant from A and 17 in. from B, show that OA will be 13 inches.

3. Prove that an angle formed by a tangent and a chord drawn through its point of contact is the supplement of any angle inscribed in the segment cut off by the chord. What is the locus of the centre of a circumference of given radius which cuts at right angles a given circumference?

4. Show that the areas of similar triangles are to each other as the squares of the homologous sides.

5. Prove that the square described upon the altitude of an equilateral triangle has an area three times as great as that of a square described upon half of one side of the triangle.

6. Find the area included between a circumference of radius 7 and the square inscribed within it.

HARVARD COLLEGE, *June*, 1893.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that two oblique lines drawn from a given point to a given line are equal if they meet the latter at equal distances from the foot of the perpendicular dropped from the point upon it.

How many lines can be drawn through a given point in a plane so as to form in each case an isosceles triangle with two given lines in the plane?

2. Prove that in the same circle, or in equal circles, equal chords are equally distant from the centre, and that of two unequal chords the less is at the greater distance from the centre.

Two chords of a certain circle bisect each other. One of them is 10 inches long; how far is it from the centre of the circle? What is the radius of the circle?

A variable chord passes, when produced, through a fixed point without a given circle. What is the locus of the middle point of the chord?

3. A common tangent of two circumferences which touch each other externally at A, touches the two circumferences at B and C respectively; show that BA is perpendicular to AC.

4. Assuming that the areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles, prove that the bisector of an angle of a triangle divides the opposite side into parts which are proportional to the sides adjacent to them.

5. Prove that the circumferences of two circles have the same ratio as their radii.

6. A quarter-mile running track consists of two parallel straight portions joined together at the ends by semicircumferences. The extreme length of the plot enclosed by the track is 180 yd. Find the cost of sodding this plot at a quarter of a dollar per sq. yd.

HARVARD COLLEGE, *June*, 1894.

1. (*a*) Prove that any quadrilateral, the opposite sides of which are equal, is a parallelogram.

(*b*) A certain parallelogram inscribed in a circumference has two sides 20 ft. in length and two sides 15 ft. in length. What are the lengths of the diagonals?

2. (*a*) Prove that if one acute angle is double the other, the triangle can be divided into two isosceles triangles by a straight line drawn through the vertex of the third angle.

(*b*) Upon a given base is constructed a triangle one of the base angles of which is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

3. (*a*) Show how to find a mean proportional between two given straight lines, but do not prove that your construction is correct.

(*b*) Prove that if from a point O in the base BC of a triangle ABC, straight lines be drawn parallel to the sides AB and AC, respectively, so as to meet AC in M and AB in N, the area of the triangle AMN is a mean proportional between the areas of the triangles BNO and CMO.

4. Assuming that the areas of two parallelograms which have an angle and a side common and two other sides unequal, but commensurable, are to each other as the unequal sides, prove that the same proportion holds good when those sides have no common measure.

5. Every cross-section of the train house of a railway station has the form of a pointed-arch made of two circular arcs the centres of which are on the ground. The radius of each arc is equal to the width of the building (210 ft.). Find the distance across the building measured over the roof, and show that the area of the cross-section is $3.675\,(4\pi - 3\sqrt{3})$ sq. ft.

HARVARD COLLEGE, *June*, 1895.

One question may be omitted.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that if two straight lines are so cut by a third that corresponding or alternate-interior angles are equal, the two lines are parallel to each other.

2. Prove that an angle formed by two chords intersecting within a circumference is measured by one-half the sum of the arcs intercepted between its sides and between the sides of its vertical angle.

Two chords which intersect within a certain circumference divide the latter into parts the lengths of which, taken in order, are as 1, 1, 2, and 5. What angles do the chords make with each other?

3. Through the point of contact of two circles which touch each other externally, any straight line is drawn terminated by the circumferences. Show that the tangents at its extremities are parallel to each other.

What is the locus of the point of contact of tangents drawn from a fixed point to the different members of a system of concentric circumferences?

4. Prove that, if from a point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and the part without the circle.

Show (without proving that your construction is correct) how you would draw a tangent to a circumference from a point without it.

5. Prove that the area of any regular polygon of an even number of sides ($2n$) inscribed in a circle is a mean proportional between the areas of the inscribed and the circumscribed polygons of half the number of sides. If n be indefinitely increased, what limit or limits do these three areas approach?

6. The perimeter of a certain church window is made up of three equal semicircumferences, the centres of which form the vertices of an equilateral triangle which has sides $3\frac{1}{2}$ ft. long. Find the area of the window and the length of its perimeter.

YALE COLLEGE, *June*, 1892.

1. Construct accurately by ruler and compass a parallelogram $ABCD$ having the angle A 45°, the base AD 6 units in length, and the altitude 3 of the same units.

Calculate the length of the line AC.

2. (*a*) State the converse of the following proposition:

If a triangle is isosceles, and if a straight line is drawn through the vertex parallel to the base, it bisects an exterior angle of the triangle.

(*b*) Prove the converse as you have stated it.

Make the demonstration as full and clear as possible.

3. Prove two of the following propositions: The work may be limited to drawing a figure and giving a synopsis of the demonstration.

(*a*) If the area of a regular polygon is equal to the product of the perimeter by one-half the apothem, it follows that the area of a circle $= \pi R^2$.

(*b*) If two lines are drawn through the same point across a circle, the products of the two distances on each line from this point to the circumference are equal to each other.

(*c*) If the radius of a circle be divided in extreme and mean ratio, the greater segment is equal to one side of a regular inscribed decagon.

YALE COLLEGE, *June*, 1893.

1. Prove that if the diagonals of a quadrilateral bisect each other the figure is a parallelogram.

2. Prove that in any right-angled triangle the square on the side opposite to the right angle is equal to the sum of the squares on the other two sides.

A purely geometrical proof is preferred.

State fully each principle employed in the proof.

3. Given a straight line AB, of indefinite length, and a point C without it. Find a point in AB equally distant from A and C.

Make the necessary construction accurately with ruler and compass.

In what case is the solution impossible?

4. Given an angle COD at the centre of a circle and the line CA meeting DO produced in A so that AB is equal to the radius of the circle. Prove that the angle A is equal to one-third of the angle COD.

YALE COLLEGE, *June*, 1894.

Geometry (*a*).

1. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

2. To draw a tangent to a given circle, so that it shall be parallel to a given straight line.

3. If AB is a chord of a circle, and CE is any chord drawn through the middle point C of the arc AB cutting the chord AB at D, prove that the chord AC is a mean proportional between CD and CE.

4. The areas of two similar triangles are to each other as the squares of any two homologous sides.

5. The area of a circle is equal to one-half the product of its circumference and radius.

YALE COLLEGE, *June*, 1894.

Geometry (*b*). [*Time* 45 *minutes.*]

1. What is the number of degrees in each angle of a regular decagon?

2. Find the area in square feet of an equilateral triangle whose side is 3 metres.

3. ABC is a right triangle. The sides AC and BC about the right angle C are respectively 50 and 120 ft. Divide the triangle into two parts equal in area by a line DF parallel to BC. Compute the length of the three sides of the triangle ADF.

4. The area of a circle is a hectare. What is its diameter?

5. Calculate in metres the length of a degree on the circumference of the earth, assuming the section of the earth to be a circle whose radius is 3963 miles. [Those taking the preliminary examinations must use logarithms.]

6. Find the value of the following expression by logarithms:

$$\sqrt[3]{\frac{(.06342)^2 \times 187.32}{.34216 \times 6.0372}}.$$

YALE COLLEGE, *June*, 1895.

Geometry (*a*).

1. (*a*) Define the terms "locus" and "limit of a variable," and give an example of each.

(*b*) Prove that two triangles are similar if their homologous sides are proportional.

(*c*) Through a given point A within a circle draw two equal chords.

[Both the construction (with ruler and compass) and also the proof are required.]

2. (*a*) Prove that if each of two angles of a quadrilateral is a right angle the bisectors of the other angles are either perpendicular or parallel to each other.

(*b*) Prove that if the radius of a circle is divided in extreme and mean ratio, the greater part is equal to the side of a regular inscribed decagon.

[The construction is not required.]

YALE COLLEGE, *June*, 1895.

One question may be omitted. Logarithmic tables should be used in calculating the answers of two questions.

Geometry (*b*).

1. The base of a triangle is 14 in. and its altitude is 7 in. Find the area of the trapezoid cut off by a line 6 in. from the vertex.

Express the result in square metres.

2. Find the number of feet in an arc of 40° 12′ if the radius of the circle is 0.7539 metres.

3. The length of a chord is 10 ft., and its greatest distance from the subtending arc is 2 ft. 7½ in. Find the radius of the circle.

4. Find the area and the weight in grams of the largest square that can be cut from a circular sheet of tin 16 in. in diameter and weighing 8.2 oz. per sq. ft.

SPECIAL AND TYPICAL QUESTIONS

1. The area of a certain regular hexagon is $294\sqrt{3}$ sq. in.; find the area and the circumference of the circumscribed circle. (Harvard, Sept., 1892.)

2. Show how to bisect the area of a triangle by a line parallel to its base, but do not take time to prove that your construction is correct. (Harvard, Sept., 1893.)

3. A stone bridge 20 ft. wide has a circular arch of 140 ft. span at the water level. The crown of the arch is $140\ (1 - \frac{1}{2}\sqrt{3})$ ft. above the surface of the water. How many square feet of surface must be gone over in cleaning so much of the under side of the arch as is above water? (*Ibid.*)

4. A kite-shaped racing track is formed by a circular arc and two tangents at its extremities. The tangents meet at an angle of 60°. The riders are to go round the track, one on a line close to the inner edge, the other on a line everywhere $5\frac{1}{4}$ ft. outside the first line. Show that the second rider is handicapped by about 22 ft. (Harvard, Sept., 1894.)

5. A and B are two fixed points on the circumference of a circle, and PQ is any diameter. What is the locus of the intersection of PA and QB? (Harvard, Sept., 1895.)

6. C is any point on the straight portion, AB, of the boundary of a semicircle. CD, drawn at right angles to AB, meets the circumference at D. DO is drawn to the centre, O, of the circle, and the perpendicular dropped from C upon OD meets OD at E. Show that DC is a mean proportional to AO and DE. (*Ibid.*)

State the fundamental theorem in the method of limits as used in Plane Geometry.

7. A horse is tethered to a hook on the inner side of a fence which bounds a circular grass plot. His tether is so long that he can just reach the centre of the plot. The area of so much of the plot as he can graze over is $\frac{98}{3}(4\pi - 3\sqrt{3})$ sq. rd.; find the length of the tether and the circumference of the plot. (*Ibid.*)

8. AB is the hypotenuse of a right triangle ABC. If perpendiculars be drawn to AB at A and B meeting AC produced at D and BC produced at E, prove the triangles ACE and BCD similar. (Geometry (*a*). Yale, Sept., 1894.)

9. To construct a square equivalent in area to a given parallelogram. (*Ibid.*)

10. Three consecutive sides of a quadrilateral inscribed in a circle subtend arcs of 82°, 99°, and 67° respectively. Find each angle of the quadrilateral in degrees, and the angle between its diagonals. (Geometry (*b*). Yale, Sept., 1894.)

11. The base of a triangle is 16 ft., and the two other sides are respectively 12 and 10 ft. Find the altitude of the triangle, and also the area. (*Ibid.*)

12. Let ABC be a right triangle. The two sides about the right angle C are respectively 455 and 1092 ft. The hypotenuse AB is divided into two segments AE and BE by the perpendicular upon it from C. Compute the lengths of AE, BE, and CE. (*Ibid.*)

13. There are two gardens: one is a square, and the other a circle; and they each contain a hectare. How much farther is it around one than the other? (*Ibid.*)

14. If the area of each is 2 hectares, what will be the difference of their perimeters? (*Ibid.*)

15. Compute by logarithms the value of

$$\sqrt{\frac{(2.3456)^3 \times (.301456)^2}{(4.02356)^4}}. \qquad (\textit{Ibid.})$$

16. (*a*) If two triangles are on equal bases and between the same parallels, a line parallel to their bases cuts off equal areas.

(*b*) Lines joining the non-adjacent extremities of two parallel chords are equal.

(*c*) State and prove the converse of the preceding proposition. (Geometry (*a*). Yale, Sept., 1895.)

17. In the triangle whose sides are a, b and c, determine the segments of each side made by the bisector of the opposite angle. (Sheffield Scientific School, June, 1893.)

18. Two triangles have sides a, b, c and a', b', c'; the angles opposite a and a' are equal; find the ratio of their areas. (Sheffield Scientific School, Sept., 1893.)

19. Construct a square whose area is 3 times that of a given square. (Sheffield Scientific School, June, 1894.)

20. The sum of the two perpendiculars dropped from any point in the base of an isosceles triangle to the equal sides is constant and equal to the perpendicular let fall from the vertex of one of the equal angles to the opposite side. (Sheffield Scientific School, Sept., 1895.)

21. Prove that if the radius be unity, the apothem of a regular inscribed decagon is $\frac{1}{4}\sqrt{10 + 2\sqrt{5}}$. (Cornell, Sept., 1891.)

22. Inscribe a square in a scalene triangle. (Cornell, Sept., 1891.)

23. Given $\frac{2}{x} = \frac{x}{3}$. Construct x. (Cornell, June, 1893.)

24. The side of an equilateral triangle inscribed in a circle is $8\sqrt{3}$. Find area of circle.

APPENDIX

LOGARITHMS

1. The **logarithm** of a number is the exponent of the power to which some assumed number must be raised to produce the given number. The assumed number is called the **base**.

Thus, assuming 10 for a base, the logarithm of 100 is 2, because the base 10 must be raised to the second power to equal 100, or if b is assumed as a base, the logarithm of b^5 is 5, because b^5 is the base b raised to the fifth power.

2. A **system of logarithms** provides the logarithms of all possible numbers, as referred to a particular base.

There are two systems of logarithms in practical use:

1. The Napierian system, used only in technical algebraic processes, in which the base $= 2.718\ 282$.

2. The common, or Briggs system, in which the base $= 10$.

By the use of the common system we are able to facilitate many numerical calculations.

3. In the common system, which alone is used in the problems here discussed, the logarithm of 10 is 1, briefly expressed, $\log 10 = 1$. Hence

$$
\begin{array}{rl}
\log 1 = 0 & [\text{since } 10^0 = 1] \\
\log 10 = 1 & [\text{ `` } 10^1 = 10] \\
\log 100 = 2 & [\text{ `` } 10^2 = 100] \\
\log 1000 = 3 & [\text{ `` } 10^3 = 1000] \\
\log .1 = -1 & [\text{ `` } 10^{-1} = \tfrac{1}{10}] \\
\log .01 = -2 & [\text{ `` } 10^{-2} = \tfrac{1}{100}] \\
\log .001 = -3 & [\text{ `` } 10^{-3} = \tfrac{1}{1000}] \\
\log .0001 = -4 & [\text{ `` } 10^{-4} = \tfrac{1}{10000}] \\
 & \text{etc.}
\end{array}
$$

These numbers are exact powers of 10, and the logarithms are integral.

The logarithm of a number which is not an exact power of 10 can be expressed only approximately, making use of an integer and a decimal. Of such a logarithm the integer is called the **characteristic** and the decimal is called the **mantissa**.

$$\log 85 = 1.92942 + \qquad \text{that is, } 10^{1.92942+} = 85$$

The characteristic of the logarithm is 1, and the mantissa is .92942.

Evidently the characteristic of all numbers from 10 to 99 inclusive is 1, and the mantissæ vary according to the particular number.

Expressed in a different way, the characteristic of a number which has one figure to the left of its decimal point is 0; of a number having two figures, it is 1; three figures, it is 2, etc.

Again, the characteristic of the logarithm of a number between 1 and .1 is -1; of a number between .1 and .01 is -2; between .01 and .001 is -3, etc.; hence

$$\log .085 = \bar{2}.92942,$$

the minus sign written over the characteristic showing that the integral part of the logarithm is negative, and the mantissa is positive. The double sign is avoided by writing such a logarithm $8.92942 - 10$.

4. The mantissæ of the logarithms of numbers are found in tables, and have been calculated by means of processes not here discussed, and the characteristics are found after the plan followed in the illustrations just given.

5. The following rules may be used:

1. *The mantissæ of the logarithms of all numbers which differ only in the position of the decimal point are the same.*

2. *The characteristic of the logarithm of a number greater than unity is* 1 *less than the number of figures preceding the decimal point.*

3. *The characteristic of the logarithm of a decimal is* 9, *minus the number of zeros between the decimal point and the first significant figure ; and* -10 *must follow the mantissa.*

For example:

$$\begin{aligned}\log\ 69745 &= 4.84351\\ \log\ 697.45 &= 2.84351\\ \log\ 6.9745 &= 0.84351\\ \log\ .069745 &= 8.84351-10\\ \log .0069745 &= 7.84351-10, \text{ etc.}\end{aligned}$$

6. USE OF LOGARITHMIC TABLES

The table of mantissæ of the logarithms of numbers between 1 and 1000 are given on pages 233–251.

The logarithm of a number, such as 54.263, is found as follows:

1. The characteristic is 1 [1 less than the number of figures preceding the decimal point], and

2. The mantissa is found by turning to the tables and finding (on page 242) under the left-hand column (headed N) the number 542; then look along to the column marked 6.

Taking care not to omit the first two significant figures of the mantissa, which are written only when these figures *change*, it is found that the mantissa corresponding to 5426 is 73448; passing to the next column, the mantissa of 5427 is 73456.

The number under discussion is between 5426 and 5427, and the mantissa of the number 5426.3 [disregarding the position of the decimal point in the given number, in finding its mantissa] is between 73448 and 73456; moreover, the increase in the mantissa, within necessary degrees of accuracy, is proportional to the increase in the number.

That is, the number 5426.3 is between 5426 and 5427, and is equal to $5426 + \frac{3}{10}$ of the difference between the two numbers. Hence, the required mantissa is between 73448 and 73456, and $= 73448 + \frac{3}{10}$ of the difference between the two mantissæ.

This difference (called the *tabular difference*) is 8, and $\frac{3}{10}$ of 8 is 2.4, and to the mantissa 73448 must be added 2, which addition or *correction* is as correct as the last significant figure itself, and the required mantissa is 73450, and the complete

$$\log 54.263 = 1.73450.$$

7. Again, find the logarithm of .00026549.

1. The characteristic is 6 [9 minus the number of zeros], − 10, and

2. The mantissa is found on page 236, and the process is illustrated in the following solution:

$$\begin{array}{rrl} \text{Mantissa of } 2654 = & 42390 & \\ & 14 & = \text{correction.} \\ \hline & 42404 & \end{array}$$

Tabular difference is 16.

$$\frac{9}{\underset{5}{\cancel{10}}} \text{ of } \overset{8}{\cancel{16}} = \frac{72}{5} = 14+$$

Ans. 6.42404 − 10.

The operation suggests the rule:

To find the logarithm of a number.

8. *Find in the tables the mantissa of the first four significant figures without regard to the position of the decimal point.*

Find the tabular difference between this mantissa and the mantissa of the next higher number of four figures.

Take such fractional part of this tabular difference as is indicated by the figures in the given number which follow the first four significant figures.

Add this correction to the mantissa first found, and

Prefix the proper characteristic.

EXAMPLES

Find the logarithms of

1. 62.44
2. 126.
3. .4592
4. 586.02
5. .043794
6. 66548.7
7. .0064391
8. .123683
9. .000024569
10. 138.4697

9. *To find the number corresponding to a given logarithm.*

For example, find the number whose logarithm is 2.23684.

The mantissa only is used to determine the significant figures of the required number, and the characteristic determines the position of the decimal point. We find in the table that the mantissa next smaller than the one given is 23679, differing from the given mantissa by 5. Of the tabular difference, 25, this difference is $\frac{5}{25}$ or $\frac{1}{5}$. Hence the required number is between 1725 (which corresponds in the table to the mantissa 23679) and 1726, and must $= 1725 + \frac{5}{25}$ (or $\frac{1}{5}$) of the difference between 1725 and 1726; that is, of unity. Expressed as a decimal, $\frac{1}{5} = .2$. Therefore the significant figures of the required number are 17252, and the characteristic 2 requires the decimal point to be between the first 2 and the 5. Therefore the number corresponding to the log 2.23684 is 172.52.

Again, find the number corresponding to the logarithm 7.34962 − 10. The process will be seen in the solution:

given mantissa = 34962
next smaller mantissa = $\underline{34947}$ corresponding number = 2236
15

tabular difference = 20
correction = $\frac{15}{20}$ of 1 = .75
number = .00223675 *Ans.*

The operations suggest the rule:

To find the number corresponding to a given logarithm.

10. *Find in the tables the next smaller mantissa, noting the four significant figures corresponding and also the tabular difference.*

Subtract from the given mantissa this next smaller mantissa.

The correction is a fraction of which the numerator is this difference and the denominator the tabular difference.

This fraction, reduced to a decimal, is affixed, omitting its decimal point, to the four significant figures first found.

Point off according to the reverse of the rules given for finding the characteristic.

EXAMPLES

Find the numbers corresponding to the following logarithms:

1. 1.32567
2. 3.98642
3. 9.88337 – 10
4. 0.37645
5. 7.11548 – 10
6. 9.22465 – 10
7. 0.98731
8. 2.00642
9. 8.55004 – 10
10. 0.12345

Treatises on algebra establish the following principles concerning logarithms:

11. *The logarithm of a product is equal to the sum of the logarithms of all its factors.*

12. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

13. *The logarithm of the power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

14. *The logarithm of a root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For example,

$$\log 144 = \log 2 + \log 9 + \log 8$$

$$\log (73 \times 131) = \log 73 + \log 131$$

$$\log \tfrac{2465}{17} = \log 2465 - \log 17$$

$$\log 32 = \log (2^5) = 5 \log 2$$

$$\log 125^{\frac{1}{3}} = \tfrac{1}{3} \log 125$$

$$\log \sqrt[12]{75} = \tfrac{1}{12} \log 75$$

$$\log \frac{15 \times 96 \times \sqrt{2}}{4\sqrt{10}} = (\log 15 + \log 96 + \tfrac{1}{2} \log 2) - (\log 4 + \tfrac{1}{2} \log 10)$$

Co-Logarithms

15. The **co-logarithm,** or arithmetical complement, of a number is the remainder obtained by subtracting its logarithm from zero.

The logarithm of 375.4 is 2.57449, and the co-logarithm of the same number is 7.42551 – 10.

The logarithm of .06819 is 8.83372 – 10, and its co-logarithm is 1.16628.

16. The process of subtracting a logarithm is the same as that of adding a co-logarithm, and the process has its chief value in the solution of fractions.

The reduction of the fraction $\frac{ab^2\sqrt[3]{c}}{2de}$ by the use of logarithms proceeds as follows: the logarithm of the result

$$= \log a + 2 \log b + \tfrac{1}{3} \log c - (\log 2 + \log d + \log e),$$

or $$= \log a + 2 \log b + \tfrac{1}{3} \log c + \text{co-log}\, 2 + \text{co-log}\, d + \text{co-log}\, e.$$

NOTE. — A *negative* quantity has no common logarithm. If negative quantities occur, disregard the minus sign, in applying the principles of logarithms, and determine the sign of the result by the ordinary algebraic processes.

EXAMPLES

Find by logarithms the values of the following:

1. $.64321 \times .043642$.
2. $(-.63249) \times .0029411$.
3. $(-.022222) \times (-.64388)^2$.
4. $(324.63)^{\frac{4}{7}} \times (6.7784)^{\frac{3}{2}}$.
5. $\sqrt[3]{\frac{-.06489}{22.477}}$.
6. $\sqrt[7]{\frac{(77777)^2}{(55555)^3}}$.
7. $\frac{\sqrt{.006742}}{-(.074329)^{\frac{3}{7}}}$.
8. $\frac{\sqrt{6.437} \times \sqrt[3]{4.7282}}{\sqrt{54.621}}$.
9. $\sqrt[3]{2} \times \sqrt[7]{3} \times \sqrt[6]{4}$.
10. $(236.47 \times 3.4267)^{\frac{1}{7}}$.
11. $\sqrt[3]{100}$.
12. $\sqrt[7]{-.05462}$.
13. $\sqrt[5]{.00010101}$.
14. $\sqrt{-3.6667}$.
15. $\sqrt[6]{\frac{13}{19}}$.
16. $\sqrt[3]{.046\sqrt{250}}$.
17. $(2\frac{13}{16})^{17}$.
18. $\sqrt[3]{\frac{3^2.2^3}{\sqrt{5}}}$.
19. $\sqrt[10]{.000001}$.
20. $(2 \times 7 \times 11 \times 191)^{\frac{4}{9}}$.
21. $\frac{\sqrt[5]{-.00543211} \times \sqrt[3]{-.47726} \times \sqrt{3246.1}}{\sqrt{642.81} \times -(.0000264)^{\frac{3}{11}}}$.
22. $(\frac{35}{1114})^{\frac{2}{7}} \div (\frac{26}{4289})^{\frac{1}{5}} \div (\frac{1}{50})^{\frac{2}{3}} \div (\frac{444}{404})^{\frac{4}{11}}$.

EXPONENTIAL EQUATIONS

1. Find x in the exponential equation $3^x = 2187$.

$$3^x = 2187$$

$$x \log 3 = \log 2187$$

$$x = \frac{\log 2187}{\log 3} = \frac{3.33985}{.47712} = 7$$

2. Find x in equation $5^x = 625$.
3. Find x in equation $3^x = 15$.
4. Find x in equation $5^x = 100$.
5. Find x in equation $mn^x = p$.
6. Find x in equation $5^{\frac{4}{x}} = 25$.

[Raise both members to the xth power.]

7. Find x in equation $\frac{ab^x + c}{d} = m$. *Ans.* $\frac{\log(dm - c) - \log a}{\log b}$.

TABLES

COMMON LOGARITHMS

OF

NUMBERS

Giving Characteristics and Mantissae of Logarithms of Numbers from 1 to 100, and Mantissae only of Numbers from 100 to 1000.

LOGARITHMS OF NUMBERS TO 100.

1	0.00000	26	1.41497	51	1.70757	76	1.88081
2	0.30103	27	1.43136	52	1.71600	77	1.88649
3	0.47712	28	1.44716	53	1.72428	78	1.89209
4	0.60206	29	1.46240	54	1.73239	79	1.89763
5	0.69897	30	1.47712	55	1.74036	80	1.90309
6	0.77815	31	1.49136	56	1.74819	81	1.90849
7	0.84510	32	1.50515	57	1.75587	82	1.91381
8	0.90309	33	1.51851	58	1.76343	83	1.91908
9	0.95424	34	1.53148	59	1.77085	84	1.92428
10	1.00000	35	1.54407	60	1.77815	85	1.92942
11	1.04139	36	1.55630	61	1.78533	86	1.93450
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13	1.11394	38	1.57978	63	1.79934	88	1.94448
14	1.14613	39	1.59106	64	2.80618	89	1.94939
15	1.17609	40	1.60206	65	1.81291	90	1.95424
16	1.20412	41	1.61278	66	1.81954	91	1.95904
17	1.23045	42	1.62325	67	1.82607	92	1.96379
18	1.25527	43	1.63347	68	1.83251	93	1.96848
19	1.27875	44	1.64345	69	1.83885	94	1.97313
20	1.30103	45	1.65321	70	1.84510	95	1.97772
21	1.32222	46	1.66276	71	1.85126	96	1.98227
22	1.34242	47	1.67210	72	1.85733	97	1.98677
23	1.36173	48	1.68124	73	1.86332	98	1.99123
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102	860	903	945	988	*030	*072	*115	*157	*199	*242	42
103	01284	326	368	410	452	494	536	578	620	662	42
104	703	745	787	828	870	912	953	995	*036	*078	42
105	02119	160	202	243	284	325	366	407	449	490	41
106	531	572	612	653	694	735	776	816	857	898	41
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302	40
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111	532	571	610	650	689	727	766	805	844	883	39
112	922	961	999	*038	*077	*115	*154	*192	*231	*269	39
113	05308	346	385	423	461	500	538	576	614	652	38
114	690	729	767	805	843	881	918	956	994	*032	38
115	06070	108	145	183	221	258	296	333	371	408	38
116	446	483	521	558	595	633	670	707	744	781	37
117	819	856	893	930	967	*004	*041	*078	*115	*151	37
118	07188	225	262	298	335	372	408	445	482	518	37
119	555	591	628	664	700	737	773	809	846	882	36
120	918	954	990	*027	*063	*099	*135	*171	*207	*243	36
121	08279	314	350	386	422	458	493	529	565	600	36
122	636	672	707	743	778	814	849	884	920	955	35
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	35
124	09342	377	412	447	482	517	552	587	621	656	35
125	691	726	760	795	830	864	899	934	968	*003	35
126	10037	072	106	140	175	209	243	278	312	346	34
127	380	415	449	483	517	551	585	619	653	687	34
128	721	755	789	823	857	890	924	958	992	*025	34
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131	727	760	793	826	860	893	926	959	992	*024	33
132	12057	090	123	156	189	222	254	287	320	352	33
133	385	418	450	483	516	548	581	613	646	678	33
134	710	743	775	808	840	872	905	937	969	*001	32
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137	672	704	735	767	799	830	862	893	925	956	32
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	31
139	14301	333	364	395	426	457	489	520	551	582	31
140	613	644	675	706	737	768	799	829	860	891	31
141	922	953	983	*014	*045	*076	*106	*137	*168	*198	31
142	15229	259	290	320	351	381	412	442	473	503	31
143	534	564	594	625	655	685	715	746	776	806	30
144	836	866	897	927	957	987	*017	*047	*077	*107	30
145	16137	167	197	227	256	286	316	346	376	406	30
146	435	465	495	524	554	584	613	643	673	702	30
147	732	761	791	820	850	879	909	938	967	997	29
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152	18184	213	241	270	298	327	355	384	412	441	29
153	469	498	526	554	583	611	639	667	696	724	28
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156	312	340	368	396	424	451	479	507	535	562	28
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158	866	893	921	948	976	*003	*030	*058	*085	*112	27
159	20140	167	194	222	249	276	303	330	358	385	27
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161	683	710	737	763	790	817	844	871	898	925	27
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192	27
163	21219	245	272	299	325	352	378	405	431	458	27
164	484	511	537	564	590	617	643	669	696	722	26
165	748	775	801	827	854	880	906	932	958	985	26
166	22011	037	063	089	115	141	167	194	220	246	26
167	272	298	324	350	376	401	427	453	479	505	26
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169	789	814	840	866	891	917	943	968	994	*019	26
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173	805	830	855	880	905	930	955	980	*005	*030	25
174	24055	080	105	130	155	180	204	229	254	279	25
175	304	329	353	378	403	428	452	477	502	527	25
176	551	576	601	625	650	674	699	724	748	773	25
177	797	822	846	871	895	920	944	969	993	*018	25
178	25042	066	091	115	139	164	188	212	237	261	24
179	285	310	334	358	382	406	431	455	479	503	24
180	527	551	575	600	624	648	672	696	720	744	24
181	768	792	816	840	864	888	912	935	959	983	24
182	26007	031	055	079	102	126	150	174	198	221	24
183	245	269	293	316	340	364	387	411	435	458	24
184	482	505	529	553	576	600	623	647	670	694	24
185	717	741	764	788	811	834	858	881	905	928	23
186	951	975	998	*021	*045	*068	*091	*114	*138	*161	23
187	27184	207	231	254	277	300	323	346	370	393	23
188	416	439	462	485	508	531	554	577	600	623	23
189	646	669	692	715	738	761	784	807	830	852	23
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191	28103	126	149	171	194	217	240	262	285	307	23
192	330	353	375	398	421	443	466	488	511	533	23
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194	780	803	825	847	870	892	914	937	959	981	22
195	29003	026	048	070	092	115	137	159	181	203	22
196	226	248	270	292	314	336	358	380	403	425	22
197	447	469	491	513	535	557	579	601	623	645	22
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202	535	557	578	600	621	643	664	685	707	728	21
203	750	771	792	814	835	856	878	899	920	942	21
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154	21
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208	806	827	848	869	890	911	931	952	973	994	21
209	32015	035	056	077	098	118	139	160	181	201	21
210	222	243	263	284	305	325	346	366	387	408	21
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214	33041	062	082	102	122	143	163	183	203	224	20
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217	646	666	686	706	726	746	766	786	806	826	20
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219	34044	064	084	104	124	143	163	183	203	223	20
220	242	262	282	301	321	341	361	380	400	420	20
221	439	459	479	498	518	537	557	577	596	616	20
222	635	655	674	694	713	733	753	772	792	811	19
223	830	850	869	889	908	928	947	967	986	*005	19
224	35025	044	064	083	102	122	141	160	180	199	19
225	218	238	257	276	295	315	334	353	372	392	19
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227	603	622	641	660	679	698	717	736	755	774	19
228	793	813	832	851	870	889	908	927	946	965	19
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154	19
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231	361	380	399	418	436	455	474	493	511	530	19
232	549	568	586	605	624	642	661	680	698	717	19
233	736	754	773	791	810	829	847	866	884	903	19
234	922	940	959	977	996	*014	*033	*051	*070	*088	18
235	37107	125	144	162	181	199	218	236	254	273	18
236	291	310	328	346	365	383	401	420	438	457	18
237	475	493	511	530	548	566	585	603	621	639	18
238	658	676	694	712	731	749	767	785	803	822	18
239	840	858	876	894	912	931	949	967	985	*003	18
240	38021	039	057	075	093	112	130	148	166	184	18
241	202	220	238	256	274	292	310	328	346	364	18
242	382	399	417	435	453	471	489	507	525	543	18
243	561	578	596	614	632	650	668	686	703	721	18
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245	917	934	952	970	987	*005	*023	*041	*058	*076	18
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253	312	329	346	364	381	398	415	432	449	466	17
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255	654	671	688	705	722	739	756	773	790	807	17
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257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	17
258	41162	179	196	212	229	246	263	280	296	313	17
259	330	347	363	380	397	414	430	447	464	481	17
260	497	514	531	547	564	581	597	614	631	647	17
261	664	681	697	714	731	747	764	780	797	814	17
262	830	847	863	880	896	913	929	946	963	979	16
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	16
264	42160	177	193	210	226	243	259	275	292	308	16
265	325	341	357	374	390	406	423	439	455	472	16
266	488	504	521	537	553	570	586	602	619	635	16
267	651	667	684	700	716	732	749	765	781	797	16
268	813	830	846	862	878	894	911	927	943	959	16
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	16
270	43136	152	169	185	201	217	233	249	265	281	16
271	297	313	329	345	361	377	393	409	425	441	16
272	457	473	489	505	521	537	553	569	584	600	16
273	616	632	648	664	680	696	712	727	743	759	16
274	775	791	807	823	838	854	870	886	902	917	16
275	933	949	965	981	996	*012	*028	*044	*059	*075	16
276	44091	107	122	138	154	170	185	201	217	232	16
277	248	264	279	295	311	326	342	358	373	389	16
278	404	420	436	451	467	483	498	514	529	545	16
279	560	576	592	607	623	638	654	669	685	700	16
280	44716	731	747	762	778	793	809	824	840	855	15
281	871	886	902	917	932	948	963	979	994	*010	15
282	45025	040	056	071	086	102	117	133	148	163	15
283	179	194	209	225	240	255	271	286	301	317	15
284	332	347	362	378	393	408	423	439	454	469	15
285	484	500	515	530	545	561	576	591	606	621	15
286	637	652	667	682	697	712	728	743	758	773	15
287	788	803	818	834	849	864	879	894	909	924	15
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289	46090	105	120	135	150	165	180	195	210	225	15
290	240	255	270	285	300	315	330	345	359	374	15
291	389	404	419	434	449	464	479	494	509	523	15
292	538	553	568	583	598	613	627	642	657	672	15
293	687	702	716	731	746	761	776	790	805	820	15
294	835	850	864	879	894	909	923	938	953	967	15
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	15
296	47129	144	159	173	188	202	217	232	246	261	15
297	276	290	305	319	334	349	363	378	392	407	15
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303	144	159	173	187	202	216	230	244	259	273	14
304	287	302	316	330	344	359	373	387	401	416	14
305	430	444	458	473	487	501	515	530	544	558	14
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307	714	728	742	756	770	785	799	813	827	841	14
308	855	869	883	897	911	926	940	954	968	982	14
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122	14
310	49136	150	164	178	192	206	220	234	248	262	14
311	276	290	304	318	332	346	360	374	388	402	14
312	415	429	443	457	471	485	499	513	527	541	14
313	554	568	582	596	610	624	638	651	665	679	14
314	693	707	721	734	748	762	776	790	803	817	14
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316	969	982	996	*010	*024	*037	*051	*065	*079	*092	14
317	50106	120	133	147	161	174	188	202	215	229	14
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319	379	393	406	420	433	447	461	474	488	501	14
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321	651	664	678	691	705	718	732	745	759	772	14
322	786	799	813	826	840	853	866	880	893	907	13
323	920	934	947	961	974	987	*001	*014	*028	*041	13
324	51055	068	081	095	108	121	135	148	162	175	13
325	188	202	215	228	242	255	268	282	295	308	13
326	322	335	348	362	375	388	402	415	428	441	13
327	455	468	481	495	508	521	534	548	561	574	13
328	587	601	614	627	640	654	667	680	693	706	13
329	720	733	746	759	772	786	799	812	825	838	13
330	851	865	878	891	904	917	930	943	957	970	13
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101	13
332	52114	127	140	153	166	179	192	205	218	231	13
333	244	257	270	284	297	310	323	336	349	362	13
334	375	388	401	414	427	440	453	466	479	492	13
335	504	517	530	543	556	569	582	595	608	621	13
336	634	647	660	673	686	699	711	724	737	750	13
337	763	776	789	802	815	827	840	853	866	879	13
338	892	905	917	930	943	956	969	982	994	*007	13
339	53020	033	046	058	071	084	097	110	122	135	13
340	148	161	173	186	199	212	224	237	250	263	13
341	275	288	301	314	326	339	352	364	377	390	13
342	403	415	428	441	453	466	479	491	504	517	13
343	529	542	555	567	580	593	605	618	631	643	13
344	656	668	681	694	706	719	732	744	757	769	13
345	782	794	807	820	832	845	857	870	882	895	13
346	908	920	933	945	958	970	983	995	*008	*020	13
347	54033	045	058	070	083	095	108	120	133	145	13
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352	654	667	679	691	704	716	728	741	753	765	12
353	777	790	802	814	827	839	851	864	876	888	12
354	900	913	925	937	949	962	974	986	998	*011	12
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356	145	157	169	182	194	206	218	230	242	255	12
357	267	279	291	303	315	328	340	352	364	376	12
358	388	400	413	425	437	449	461	473	485	497	12
359	509	522	534	546	558	570	582	594	606	618	12
360	630	642	654	666	678	691	703	715	727	739	12
361	751	763	775	787	799	811	823	835	847	859	12
362	871	883	895	907	919	931	943	955	967	979	12
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098	12
364	56110	122	134	146	158	170	182	194	205	217	12
365	229	241	253	265	277	289	301	312	324	336	12
366	348	360	372	384	396	407	419	431	443	455	12
367	467	478	490	502	514	526	538	549	561	573	12
368	585	597	608	620	632	644	656	667	679	691	12
369	703	714	726	738	750	761	773	785	797	808	12
370	820	832	844	855	867	879	891	902	914	926	12
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372	57054	066	078	089	101	113	124	136	148	159	12
373	171	183	194	206	217	229	241	252	264	276	12
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375	403	415	426	438	449	461	473	484	496	507	12
376	519	530	542	553	565	576	588	600	611	623	12
377	634	646	657	669	680	692	703	715	726	738	11
378	749	761	772	784	795	807	818	830	841	852	11
379	864	875	887	898	910	921	933	944	955	967	11
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081	11
381	58092	104	115	127	138	149	161	172	184	195	11
382	206	218	229	240	252	263	274	286	297	309	11
383	320	331	343	354	365	377	388	399	410	422	11
384	433	444	456	467	478	490	501	512	524	535	11
385	546	557	569	580	591	602	614	625	636	647	11
386	659	670	681	692	704	715	726	737	749	760	11
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405	746	756	767	778	788	799	810	821	831	842	11
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407	959	970	981	991	*002	*013	*023	*034	*045	*055	11
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671	672	679	685	692	698	705	711	718	724	730	6
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675	930	937	943	950	956	963	969	975	982	988	6
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052	6
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678	123	129	136	142	149	155	161	168	174	181	6
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693	073	080	086	092	098	105	111	117	123	130	6
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696	261	267	273	280	286	292	298	305	311	317	6
697	323	330	336	342	348	354	361	367	373	379	6
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949	727	731	736	740	745	749	754	759	763	768	5
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
950	772	777	782	786	791	795	800	804	809	813	5
951	818	823	827	832	836	841	845	850	855	859	5
952	864	868	873	877	882	886	891	896	900	905	5
953	909	914	918	923	928	932	937	941	946	950	5
954	955	959	964	968	973	978	982	987	991	996	5
955	98000	005	009	014	019	023	028	032	037	041	5
956	046	050	055	059	064	068	073	078	082	087	5
957	091	096	100	105	109	114	118	123	127	132	5
958	137	141	146	150	155	159	164	168	173	177	5
959	182	186	191	195	200	204	209	214	218	223	5
960	227	232	236	241	245	250	254	259	263	268	5
961	272	277	281	286	290	295	299	304	308	313	5
962	318	322	327	331	336	340	345	349	354	358	5
963	363	367	372	376	381	385	390	394	399	403	5
964	408	412	417	421	426	430	435	439	444	448	5
965	453	457	462	466	471	475	480	484	489	493	4
966	498	502	507	511	516	520	525	529	534	538	4
967	543	547	552	556	561	565	570	574	579	583	4
968	588	592	597	601	605	610	614	619	623	628	4
969	632	637	641	646	650	655	659	664	668	673	4
970	677	682	686	691	695	700	704	709	713	717	4
971	722	726	731	735	740	744	749	753	758	762	4
972	767	771	776	780	784	789	793	798	802	807	4
973	811	816	820	825	829	834	838	843	847	851	4
974	856	860	865	869	874	878	883	887	892	896	4
975	900	905	909	914	918	923	927	932	936	941	4
976	945	949	954	958	963	967	972	976	981	985	4
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	4
978	99034	038	043	047	052	056	061	065	069	074	4
979	078	083	087	092	096	100	105	109	114	118	4
980	123	127	131	136	140	145	149	154	158	162	4
981	167	171	176	180	185	189	193	198	202	207	4
982	211	216	220	224	229	233	238	242	247	251	4
983	255	260	264	269	273	277	282	286	291	295	4
984	300	304	308	313	317	322	326	330	335	339	4
985	344	348	352	357	361	366	370	374	379	383	4
986	388	392	396	401	405	410	414	419	423	427	4
987	432	436	441	445	449	454	458	463	467	471	4
988	476	480	484	489	493	498	502	506	511	515	4
989	520	524	528	533	537	542	546	550	555	559	4
990	564	568	572	577	581	585	590	594	599	603	4
991	607	612	616	621	625	629	634	638	642	647	4
992	651	656	660	664	669	673	977	682	686	691	4
993	695	699	704	708	712	717	721	726	730	734	4
994	739	743	747	752	756	760	765	769	774	778	4
995	782	787	791	795	800	804	808	813	817	822	4
996	826	830	835	839	843	848	852	856	861	865	4
997	870	874	878	883	887	891	896	900	904	909	4
998	913	917	922	926	930	935	939	944	948	952	4
999	957	961	965	970	974	978	983	987	991	996	4
N.	0	1	2	3	4	5	6	7	8	9	D.

METRIC SYSTEM TABLES

LINEAR MEASURE

10 millimeters (^{mm}) = 1 centimeter (^{cm})
10 centimeters = 1 decimeter (^{dm})
10 decimeters = 1 *meter* (^{m})
10 meters = 1 dekameter (^{Dm})
10 dekameters = 1 hectometer (^{Hm})
10 hectometers = 1 kilometer (^{Km})

SQUARE MEASURE

100 sq. mm. = 1 sq. cm.
100 sq. cm. = 1 sq. dm.
100 sq. dm. = 1 sq. m. = 1 centare (^{ca})
100 sq. m. = 1 sq. Dm. = 1 are (^{a})
100 sq. Dm. = 1 sq. Hm. = 1 hectare (^{Ha})
100 sq. Hm. = 1 sq. Km.

CUBIC MEASURE

1000 cu. mm. = 1 cu. cm.
1000 cu. cm. = 1 cu. dm.
1000 cu. dm. = 1 cu. m.

CAPACITY MEASURE

1 milliliter (^{ml}) = 1 cu. cm.
10 milliliters = 1 centiliter (^{cl})
10 centiliters = 1 deciliter (^{dl})
10 deciliters = 1 *liter* (^{l}) = 1 cu. dm.
10 liters = 1 dekaliter (^{Dl})
10 dekaliters = 1 hectoliter (^{Hl})
10 hectoliters = 1 kiloliter (^{Kl}) = 1 cu. m.

WEIGHTS

10 milligrams ($^{\text{mg}}$) = 1 centigram ($^{\text{cg}}$)
10 centigrams = 1 decigram ($^{\text{dg}}$)
10 decigrams = 1 *gram* ($^{\text{g}}$)
10 grams = 1 dekagram ($^{\text{Dg}}$)
10 dekagrams = 1 hectogram ($^{\text{Hg}}$)
10 hectograms = 1 kilogram or kilo ($^{\text{K}}$)
1000 kilograms = 1 ton ($^{\text{T}}$)

KEY TO SYSTEM

1 cu. dm. measures 1 liter and (pure water, at 4° centigrade) weighs 1 kilo.

METRIC EQUIVALENTS

1 meter = 39.37 in. = 1.0936 yd.
1 hectare = 2.47 acres
1 liter $= \begin{cases} .908 \text{ qt. dry} \\ 1.0567 \text{ qt. liq.} \end{cases}$
1 kilogram = 2.2046 lbs.

1 yard = 0.9144 m.
1 acre = 0.4047 ha.
1 qt. dry = 1.101 l.
1 qt. liq. = 0.9463 l.
1 pound = 0.4536 K.

APPROXIMATE METRIC EQUIVALENTS

1 dm. = 4 in.
1 m. = 40 in.
1 km. = $\frac{5}{8}$ mile
1 ha. = $2\frac{1}{2}$ acres

1 l. $= \begin{cases} \frac{17}{16} \text{ liq. qt.} \\ \frac{9}{10} \text{ dry qt.} \end{cases}$
1 Hl. = $2\frac{5}{6}$ bush.
1 K. = $2\frac{1}{5}$ lbs.
1 T. = 2200 lbs.

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